Stochastic Economic Growth: An Operator-Theoretic Approach

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Abstract: For many years the trend in macroeconomics has been towards models which are both explicitly stochastic and explicitly dynamic. With these models, researchers seek to replicate and explain observable properties of the major economic time series. One manifestation of this trend towards stochastic dynamic modeling has been increasing use of the inherently dynamic models developed in the field of economic growth. The latter have proved to be suitable not only for the study of growth and development, but also for that of many other areas within macroeconomics, such as business cycles, fiscal policy and public finance.

This thesis is a re-examination of the stochastic dynamics arising from some well-known models of economic growth. The focus is particularly on ergodic properties and the existence of stable equilibria, where equilibrium here is defined in the usual stochastic growth sense (i.e., as the stationary distribution of a Markov process). Brief consideration is also given to asymptotic statistical properties of economic time series, such as law of large numbers and central limit tendencies.

On one hand, the thesis has been motivated by the availability of new and unexploited techniques for studying the kinds of Markovian systems generated by these models. The techniques in question are operator-theoretic, with a particular focus on integral Markov semigroups in the function space L_1. They are particularly well suited to analysis of Markov chains on unbounded state space.

On the other hand, motivation also comes from the demand side: new conditions for evaluating the stability of stochastic dynamic models are valuable

to economists who are not familiar with recent mathematical innovations. In this connection, the thesis has sought to provide sufficient conditions that are easy to verify in applications and admit standard models in the econometric tradition.

A highlight of the thesis is a set of new sufficient conditions for the stability of perturbed dynamical systems on the nonnegative half-ray $\mathbb{R}_+$. By introducing a Lyapunov criterion, a set of general conditions is found which includes existing work from the mathematical literature as a special case.

This is to certify that

- the thesis comprises only my original work towards the PhD,
- due acknowledgement has been made in the text to all other material used,
- the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

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Contents

Chapter 1

Introduction

1.1 Overview of the Thesis

The stochastic growth model, which links production to savings, and savings in turn to investment and production, has become a benchmark model in economics, not just for theoretical and empirical growth problems, but also for areas ranging from business cycle theory and finance to public sector economics and development. This thesis presents four essays which examine the dynamic properties of a collection of well-known growth models. The primary contribution is new sufficient conditions for existence, uniqueness and stability of stochastic equilibria. A small amount of supplementary material considers related dynamic properties, such as the law of large numbers and the central limit theorem.

The theme which unites the essays is our focus on operator-theoretic methods. Growth models are viewed as Markov chains, which in turn are translated into deterministic linear operators sending a space of signed measures into itself. These maps—the so-called *Markov operators*—are defined such that their iteration on some initial condition generates the time-indexed sequence of marginal distributions for the state variables in the growth model. The primary objective of the studies is to obtain conditions such that this sequence of distributions converges to a unique invariant limit, the latter being independent of the economy's initial conditions.

Many of the arguments are based on a framework for proving global stability of Markov operators suggested by the Polish mathematician Andrzej Lasota [38]. Lasota implements a straightforward but relatively unknown fixed point technique, which combines an interesting contraction condition with a compactness notion called Lagrange stability.

1.1.1 Structure of the Thesis

Following this introduction, the thesis is divided into five main chapters (Chapters 2–6). The first of these (Chapter 2) is an exposition of the mathematical framework outlined in the previous paragraph. The objective is to set out the key ideas and definitions, in order to avoid having to repeat them in each of the remaining theory chapters. Some of the results in this preliminary chapter are—to the best of the author's knowledge—new, and the key proofs of existing results are also new.

The preliminary chapter is followed by four essays on stochastic growth theory. The first essay (Chapter 3, based on the manuscript "Stochastic Optimal Growth with Unbounded Shock," forthcoming in the *Journal of Economic*

Theory, doi: 10.1006/jeth.2001.2842) treats the by-now classic problem of Brock and Mirman [10]. Our study extends the existing analysis to a new class of productivity shocks—including those common to econometric modeling. It is hoped that this extension will therefore be helpful in further integrating theoretical and empirical research which is based on the Brock-Mirman framework.

Chapter 4—perhaps the core contribution of the thesis—sets out a new framework for determining existence, uniqueness and stability of equilibria in economic systems evolving on the nonnegative real numbers. While this essay is in essence an applied mathematics paper that has applications to many areas other than growth theory (and indeed other sciences), the research was motivated by the author's investigations into the stochastic growth problem, particularly the Brock-Mirman problem that was the subject of the first essay. Moreover, it has obvious applications to a range of one-sector growth models, whether they be of the Ramsey-Brock-Mirman, overlapping generations or Solow-Swan formulation.

When the distributions of the state variables in a Markov chain converge over time to a unique limiting distribution, one can perhaps anticipate that if, for example, one takes a sufficiently long time series from that model and calculates the average value, then that value should be close to the mean of the limiting distribution. The third essay (Chapter 5, based on the manuscript "Stochastic Growth: Asymptotic Distributions," forthcoming in the journal *Economic Theory*) focuses on these kinds of econometric questions. More precisely, we seek sufficient conditions for globally stable stochastic growth models to also have law of large numbers and central limit theorem properties. Again our treatment admits the standard econometric shocks. Instead of restricting the support of shocks, we require a rather strong "average contraction" condition on technology.

Finally, the fourth essay (Chapter 6) discusses linearization of random systems. Linearization is a common approach to nonlinear systems, as the dynamics of linear stochastic models are well-understood. However, this approach uses a potentially flawed logic; there is no general result which links the dynamic properties of the derived linear model with those of the true (nonlinear) model. In this chapter, operator-theoretic techniques are used to justify linearization in the important special case of log-linear models.

1.1.2 Sufficient Conditions

As a general comment regarding sufficient conditions for mathematical properties, there are two main criteria against which these conditions must be assessed. The first is generality—to what extent do the sufficient conditions *characterize* the set of models from within a given class which have the property in question. The second criterion is the ease with which the conditions can be verified from the primitives of a particular model.

These two criteria are in a sense conflicting. For example, a very general sufficient condition for a given model to have property P is property P. Obviously the condition is sufficient (P implies P). Moreover, the condition characterizes the class of models that have property P (not P implies not P).

Yet it is clear that this sufficient condition does not give the researcher who seeks to verify property P in a specific model much additional information.

The objective of the author has been to balance the two criteria. On one hand, the conditions contained here often extend existing work. On the other hand, it is hoped that the conditions will prove easy for applied researchers to verify, being stated in a mathematical language which is familiar to the non-theorist.

1.2 Existing Literature

Extensive literature reviews are included at the beginning of each essay. However, the thesis as a whole was particularly influenced by the following work.

On the economic side, the most important reference was without doubt the classic stochastic optimal growth paper of Brock and Mirman [10]. In addition, Mirman published a number of papers around the same time which all focused on the dynamic properties of stochastic neoclassical economies with convex (i.e., decreasing returns) technology [44, 45].

In attempting to extend these initial contributions, current theorists are bestowed with Gerschenkron's advantage of backwardness. It is sometimes difficult to fully acknowledge—or even remain conscious of—the insights that were laid out in front of us for the first time by those earlier authors.

On the mathematical side, the greatest influence on this thesis has been the work of Andrzej Lasota, a functional analyst at the Polish Academy of

Science. In particular, the author has made extensive use of Lasota's invariant principle for Markov semigroups [38], as well as various ideas from his monograph on the L_1 approach to dynamic systems with M. C. Mackey [40].

Chapter 2

Equilibria of Markovian Models

2.1 Introduction

Increasingly, modern economics in general and growth theory in particular is implemented within the framework of stochastic dynamic systems. Physical laws, equilibrium constraints and restrictions on the behavior of agents jointly determine evolution of endogenous state variable $x \in X$ according to some transition rule

$$x_{t+1} = S(x_t, z_t, \varepsilon_t), \quad t = 0, 1, \ldots, \tag{2.1}$$

where S is an arbitrary function, (z_t) is a sequence of exogenous forcing variables and (ε_t) is uncorrelated noise.

For some models, either z_t is constant or the endogenous variables can be conveniently redefined such that the system is autonomous:

$$x_{t+1} = T(x_t, \varepsilon_t), \quad t = 0, 1, \ldots \tag{2.2}$$

Assume that this is the case.

Of primary concern is whether the autonomous system (2.2) is in some sense stationary, in which case one can anticipate convergence of the sequence of distributions (φ_t) associated with the sequence of random variables (x_t) to some unique limiting distribution φ^*. The latter is then interpreted as the long-run equilibrium of the economy (2.2). Typically, comparative dynamics (policy simulation) will be performed by analyzing the relationship between its moments and the underlying structural parameters contained in the function T and the distribution of the shock ε.

When T is linear on real vector space, (2.2) is the standard autoregression (AR) model. Conditions for stationarity are familiar from elementary time series analysis [25]. When the map is nonlinear, dynamic behavior is potentially more complicated. General conditions for existence of unique and stable equilibria are not known.

In this case, a common approach in the applied literature is to linearize (2.2) using a first order Taylor expansion or similar technique, and then examine the stability properties of the resulting AR model. However, it is by no means clear that stability properties obtained for the AR model have *any* homeomorphic implications for the behavior of the true model (2.2). In other words, it is not in general legitimate to infer stability of (2.2) from stability of the corresponding linear form. Moreover, linearization may eliminate important features of the model.[1]

[1]For example, Durlauf and Quah [16] find evidence to the effect that standard linearization procedures applied to Solow-Ramsey growth models fail to extract nonlinear

A more correct method is to examine the Markov chain generated by (2.2), and determine whether appropriate conditions for stability of Markovian systems are satisfied. Early examples are Brock and Mirman [10], Mirman [44, 45], Green and Majumdar [24], Brock and Majumdar [9] and Razin and Yahav [51]. An excellent survey of sufficient conditions is provided by Futia [21]. Stokey, Lucas and Prescott [58, Chapter 13] outline ways to verify these and related conditions for common economic models. Prescott and Hopenhayn [28] develop new sufficient conditions using only monotonicity and a mixing condition. Bhattacharya and Majumdar [7] obtain exponential convergence in the Kolmogorov metric for real-valued systems that satisfy a "splitting" condition.

In this thesis we treat precisely the same problem, but use a different set of techniques that have evolved recently in the applied mathematics, physics and biology literature. In this preliminary chapter, a rather detailed introduction to these mathematical techniques is given.

2.1.1 Growth and Markov Chains

As the thesis treats only sequences of uncorrelated shocks, growth models in the form of (2.2) can be reduced to Markov chains. While there are many approaches to the study of Markov chains, we find the operator-theoretic framework most applicable to solving the relevant economic problems. Within this framework our focus is primarily on the L_1 method.

local increasing returns dynamics that are critical to understanding the evolution of the cross-country income distribution.

Much of the early L_1 theory is due to Hopf [29]. The monograph of Foguel [20] contains an extensive survey of asymptotic results. Lasota and Mackey [40] use L_1 techniques to study perturbed and chaotic systems. Operator-theoretic treatment of Markov processes begins with Krylov and Bogolioubov [35]. See also Kakutani and Yoshida [33]. Recently the most active work within the L_1 framework is that being conducted by a team of Polish mathematicians led by A. Lasota [38].[2]

Recently, L_1 methods have been applied to the study of particle energy in an ideal gas [38], fluctuations in the brightness of the Milky Way [40], propagation of annual plants with seed-bank [30], and cell growth in a proliferating cell population [39, 61, 60, 41, 38].

2.1.2 Outline of the Chapter

Section 2.2 gives a heuristic overview of how perturbed dynamical systems in general and stochastic growth models in particular can be rewritten as continuous linear operators called Markov operators. In Section 2.3 this discussion is formalized and related to the existing economic literature.

Performing the transformation into Markov operators converts the equilibrium problem into one of finding fixed points for such operators. Section 2.4 discusses useful fixed point arguments for linear operators on metric and topological vector spaces. Section 2.5 shows how the general fixed point results from Section 2.4 can be applied to certain types of Markov operators.

[2] See also Lasota and Mackey [40] and Horbacz [30].

2.2 Overview of the Method

This section gives a broad and heuristic overview of the mathematical techniques used in the thesis. The discussion is intended to be intuitive rather than rigorous. In particular, no definition of integral is given, and mathematical objects are referred to without attempting to prove that they do in fact exist. Formal arguments begin in Section 2.3.

2.2.1 Outline

For dynamic economic models, an equilibrium (or steady state) is defined to be a point in the state space that is stationary under the period-to-period transition rule. If such a point is obtained, then no further change is observed in the system. As well as this invariance property, equilibria may be attractive for points in the surrounding state space, which is to say that the transition rule moves nearby points closer to the equilibrium.

In the case of stochastic models, a state cannot be stationary in the same sense as those in deterministic systems, given that shocks continue to disturb activity in each period. Instead, a steady state must be viewed as a situation where the probabilistic laws that govern the state variables cease to change over time [24]. For stochastic economies the notion of stable equilibrium can be approached as follows. Since the path of the economy is a stochastic process, the state at any time in the future can be known only up to a probability distribution. Hence the state space is re-interpreted to be the collection of all density functions on the original space. Densities can be identified with points on the unit sphere in the space of integrable functions. (The set of densities coincides with the intersection of the positive cone and the boundary of the unit sphere.) Thus any stable stochastic equilibrium can be viewed as a point on this infinite dimensional sphere to which nearby points are attracted as time evolves.

In this sense, deterministic and stochastic equilibria can be thought of as differing not conceptually but rather in the nature (in particular, in the dimension) of the space in which they are located. Here the above identification of stochastic equilibria with attractors on the unit sphere of the space of integrable functions is exploited to obtain sufficient conditions for the existence of stable equilibria in a range of stochastic growth models.

2.2.2 Discrete Dynamical Systems

Consider first a deterministic abstract system characterized at each time t by a vector of state variables x_t taking values in state space X. Evolution is governed by a first-order difference equation

$$x_{t+1} = Tx_t, \quad T\colon X \to X. \tag{2.3}$$

The map T encodes the structure of the economic system, which is in turn determined by the primitives of the model, such as preferences, technology and market conditions. A realization or trajectory for the system is a sequence $(T^n x)$ in X generated by iterating the map T on initial state x.[3] An equilibrium is a fixed point of T on X.

[3] Here $T^n x$ is defined recursively by $T^n x = T(T^{n-1}x)$, $T^1 x = Tx$.

More generally, a *semidynamical system* is a pair (X, T), where X is a topological space and T is a continuous mapping of X into itself.[4] An *equilibrium* or *steady state* of (X, T) is a fixed point of T on X (i.e., a point $x^* \in X$ such that $Tx^* = x^*$). Fixed points are said to be *stationary* or *invariant* under T. Similar terminology also applies to sets. If $TA \subset A$, then A is said to be *invariant* under T. For fixed point x^* of T on X, the *stable set* $S_T(x^*)$ of x^* is that subset of X which is convergent to x^* under iteration of T:[5]

$$S_T(x^*) = \{x \in X \; : \; T^n x \to x^* \;\; (n \to \infty)\}.$$

The point x^* is said to be *stable*, or an *attractor*, whenever there exists a set G open in X such that $x^* \in G$ and $S_T(x^*) \supset G$. In particular:

Definition 2.1. Semidynamical system (X, T) is said to be *globally asymptotically stable* (or just *globally stable*) if there exists a unique fixed point x^* and $S_T(x^*) = X$.

Figure 2.1 shows motion induced by iteration of an arbitrary map T, $X = \mathbb{R}^2$. Continued iteration generates a sequence in the plane.

[4] The system is called *dynamical* if, in addition, the mapping T is invertible with continuous inverse (i.e. is a homeomorphism).

[5] In the interests of generality, X has been defined to be a topological space. However, the reader may think of it as having sufficient structure to make convergence of sequences a significant concept. For example, we may imagine throughout that X is a Hausdorff space.

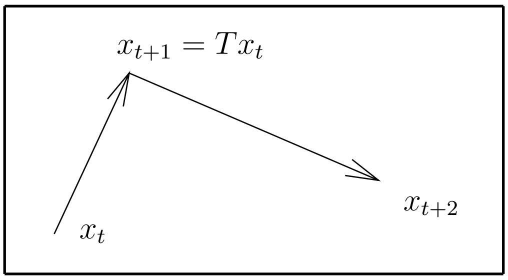

Figure 2.1: Deterministic system in $\mathbb{R}^2$.

2.2.3 Perturbed Dynamical Systems

Suppose now that the system (2.3) is perturbed at each transition from state x_t to state x_{t+1} by serially uncorrelated, X-valued shock ε_t with distribution given by density ψ:

$$x_{t+1} = T(x_t, \varepsilon_t), \quad \varepsilon_t \sim \psi, \quad T\colon X \times X \to X. \tag{2.4}$$

For each fixed $x_t \in X$, x_{t+1} is a random variable with distribution uniquely determined by the value of x_t, the density ψ and the map T. Let the density of this conditional distribution be $p(x_t, \cdot)$. That is,

$$p\colon X \times X \to \mathbb{R}, \quad \mathrm{Prob}(x_{t+1} \in B | x_t) = \int_B p(x_t, x_{t+1}) dx_{t+1}, \tag{2.5}$$

where $\mathrm{Prob}(x_{t+1} \in B | x_t)$ is the probability that the state vector is in $B \subset X$ at time $t+1$ given its current location at x_t. Figure 2.2 shows a perturbed system with additive shock in state space $\mathbb{R}^2$. The circles represent contour lines for the conditional density $p(x_t, \cdot)$. The bold arrows are sample realizations of the process.

The formulation (2.5) is convenient for calculation of the *unconditional* distribution of the state vector at each point in time. Suppose that the uncon-

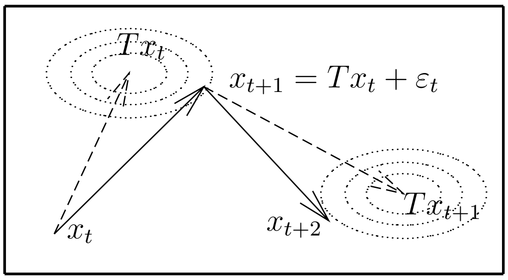

Figure 2.2: The perturbed system.

ditional (marginal) distribution of x_t is known, and is given by density φ_t. In this case,

$$\varphi_{t+1}(x_{t+1}) = \int p(x_t, x_{t+1})\varphi_t(x_t)dx_t \tag{2.6}$$

defines the unconditional density of the state at time $t + 1$. The intuition is that the integral sums the probability $p(x_t, x_{t+1})$ of traveling to x_{t+1} from x_t for all $x_t \in X$, weighted at each point by the likelihood $\varphi_t(x_t)$ of x_t occurring as the current state. The recursion (2.6) provides a way to calculate the entire sequence of densities (φ_t) that represent the marginal distributions for the stochastic process (x_t) from any initial density φ_0 ($x_0 \sim \varphi_0$).

In analyzing the behavior of the sequence (φ_t), one possibility is to use standard techniques from the classical theory of Markov processes.[6] However, it is also possible to frame the same problem as a semidynamical system. The idea is to re-interpret the state space to be the collection of all densities on X. Call this set D. The other half of the pair is the operator (call it P) that

[6]See, for example, Shiryaev [52, Chapter 8].

associates current-period with next-period densities through the integration defined in (2.6).

In this notation, (2.6) can be rewritten as

$$\varphi_{t+1} = P\varphi_t, \quad P\colon D \to D. \tag{2.7}$$

But the recursion (2.7) is now in exactly the same formula as the deterministic system (2.3), which means that similar techniques can be applied to its analysis. This translation of the perturbed system (2.4) into a deterministic map on the space of density functions is called the L_1 approach to Markov processes. Evolution of the economy is characterized by a sequence of densities generated by iterating P on some initial density φ_0. An equilibrium is a fixed point of the semidynamical system (D, P). The economy has a unique, globally stable equilibrium whenever (D, P) is asymptotically stable in the sense of Definition 2.1, 21.

These definitions are consistent with those used in previous studies.[7] However, the space of possible states D and hence equilibria has been constructed to include only those distributions that can be represented by density functions. Thus probability mass cannot be concentrated at a point. In particular, this means that the model does not include the deterministic system as a special case; the distribution of the disturbance term ε must be nondegenerate.

[7] The operator P is analogous to T^* in Brock and Majumdar [9, Eq. (4.3)], Futia [21, p. 380], and Stokey et al. [58, Eq. (2), p. 213], and to T in Hopenhayn and Prescott [28, p. 1392].

2.3 Markov Operators

A more formal discussion of perturbed dynamical systems and Markov operators is now given. To begin, let X be a topological space, let $\mathcal{B}$ be the Borel sets of X and let μ be a fixed σ-finite measure on $(X, \mathcal{B})$. In what follows, density functions will be defined in terms of their integral with respect to μ. Integration where the measure is not made explicit is taken with respect to μ; integration using the symbol $\int$ without subscript is taken over the whole space X.

Let $\mathcal{M}$ be the normed vector lattice of finite signed Borel measures on X with standard partial order and total variation norm.[8] Let $\mathcal{P}$ be the elements $\nu \in \mathcal{M}$ such that $\nu \geq 0$ and $\nu(X) = \|\nu\| = 1$. The subset $\mathcal{P}$ will be called the *distributions* on X.

Further, let $L_1(\mu)$ be the space of μ-integrable real functions on the measurable space $(X, \mathcal{B})$. As usual, $L_1(\mu)$ is interpreted as a Banach lattice of equivalence classes; functions equal off a μ-null set are identified. A *density function* on X is an element $\varphi \in L_1(\mu)$ such that $\varphi \geq 0$ and $\int \varphi = \|\varphi\| = 1$. The set of all density functions is denoted $D(\mu)$.

The sets $\mathcal{M}$ and $L_1(\mu)$ are related in that $L_1(\mu)$ is isometrically and lattice isomorphic to that subset of $\mathcal{M}$ (call it $\mathcal{M}_\mu$) which is absolutely continuous with respect to μ. The relevant isomorphism is Radon-Nikodým (RN) differentiation. In what follows the two sets $L_1(\mu)$ and $\mathcal{M}_\mu$ are identified; we do

[8] In other words, $\mathcal{M}$ is the class of countably additive real functions on $\mathcal{B}$. For a definition of the total variation norm, see, for example, Stokey et al. [58, Section 11.3], or Futia [21, p. 380].

not distinguish between them in the presentation.

2.3.1 Operators on Measures

Consider again the economic model defined by (2.4). Random outcomes (states of nature) are implemented as follows. For some measurable space $(\Omega, \mathcal{F})$, and for some fixed probability measure $\mathbf{P}$ on $(\Omega, \mathcal{F})$, a state of nature is selected from Ω according to $\mathbf{P}$, and mapped into X by random variable $\varepsilon \colon \Omega \to X$. The random variable defines a distribution $\Psi \in \mathcal{P}$ associating event $B \in \mathcal{B}$ with the real number $\mathbf{P}[\varepsilon^{-1}(B)] \in [0, 1]$.

In what follows, by a *stochastic dynamic economy on* X is meant a pair (T, Ψ), where T is a map from $X \times X$ into X, and, given current state value $x_t \in X$, a shock $\varepsilon_t \in X$ is selected independently from Ψ, and the next period state is realized as

$$x_{t+1} = T(x_t, \varepsilon_t).$$

Dynamics of Markovian models are usually described in terms of transition kernels [21, Definition 1.1]. Let $\mathbf{1}_B \colon X \to \{0, 1\}$ be the characteristic function for $B \in \mathcal{B}$.[9] The economy (T, Ψ) determines a Markov process on X with transition kernel $\mathbf{N}$,

$$\mathbf{N} \colon X \times \mathcal{B} \ni (x, B) \mapsto \int \mathbf{1}_B[T(x, z)]\Psi(dz) \in [0, 1]. \tag{2.8}$$

The value $\mathbf{N}(x, B)$ should be interpreted as the conditional probability that the next period state is in Borel set B, given that the current state is equal to x. A Markov process is fully characterized by its transition kernel.

[9]Thus $\mathbf{1}_B(x) = 1$ when $x \in B$ and zero otherwise.

We seek to derive using $\mathbf{N}$ a recursion that links successive marginal distributions of the state variables. Let B be any Borel set, and let $\nu_t \in \mathcal{P}$ be the marginal distribution for the random variable x_t.[10] By the law of total probability, if ν_{t+1} is the distribution for x_{t+1}, then

$$\nu_{t+1}(B) = \int \mathbf{N}(x, B)\nu_t(dx). \tag{2.9}$$

Intuitively, the probability that the state variable is in B next period is the sum of the probabilities that it travels to B from x across all $x \in X$, weighted by the probability $\nu_t(dx)$ that x occurs as the current state.

Following Futia [21], Stokey et al. [58] and other authors, the relationship (2.9) is redefined in terms of operators. Suppose we define an operator $P\colon \mathcal{M} \ni \nu \mapsto P\nu \in \mathcal{M}$ by

$$P\nu(B) = \int \mathbf{N}(x, B)\nu(dx). \tag{2.10}$$

It follows from (2.9) and (2.10) that if ν_t is the distribution for the current state x_t, then $P\nu_t$ is the distribution for the next period state x_{t+1}.

Repeated iteration of P on a fixed distribution ν is equivalent to moving forward in time. If P^t is defined by $P^t = P \circ P^{t-1}$ and $P^1 = P$, and if ν is the current marginal distribution for the state variable, then $P^t\nu$ is the distribution t periods hence.

Evidently P is linear and also positive, in the sense that it maps the positive cone of $\mathcal{M}$ (i.e., the finite measures on $\mathcal{B}$) into itself. In addition, $P\mathcal{P} \subset \mathcal{P}$,

[10] The distribution for the entire stochastic process $(x_t)_{t\geq 0}$ can be constructed uniquely from the transition kernel and an initial value x_0 [52, Theorem II.9.2]. The real number $\nu_t(B)$ is the probability that this distribution assigns to the event $x_t \in B$ and $x_s \in X$ for all other $s \neq t$.

because if $\nu \in \mathcal{P}$, then $\nu(X) = 1$, and hence $P\nu(X) = \int N(x, X)\nu(dx) = \nu(X) = 1$.

At this point we introduce the notion of a Markov operator. Our definition is a generalization of Hopf [29, Definition 2.1].

Definition 2.2. Let U be an ordered normed space. Let $D \subset U$ be the intersection of the positive cone and the boundary of the unit sphere in U. A *Markov operator* on U is any linear operator $Q\colon U \to U$ such that $QD \subset D$.[11]

It is well-known that every positive linear operator from a Banach lattice into itself is automatically continuous [1, Theorem 8.6]. Evidently Markov operators are positive. Thus, when U is a Banach lattice, every Markov operator is automatically norm-continuous on all of U by positivity. In this case, it is clear that (D, Q) forms a semidynamical system in the sense of the definition given in Section 2.2.2, page 20.

By the above discussion, the operator P defined in (2.10) is a Markov operator on $\mathcal{M}$. The following notion of global stability corresponds to our earlier definition of stability for semidynamical systems (Definition 2.1, p. 21) as applied to $(\mathcal{P}, P)$.

Definition 2.3. Let (T, Ψ) be a perturbed dynamical system. Let P be the corresponding Markov operator. An *equilibrium* or *steady state* for (T, Ψ) is a distribution $\nu^* \in \mathcal{P}$ such that $P\nu^* = \nu^*$. An equilibrium ν^* is called *unique* if there exists no other fixed point of P in the space $\mathcal{P}$, and *globally stable* if $P^t\nu \to \nu^*$ in the total variation norm as $t \to \infty$ for every $\nu \in \mathcal{P}$.

[11] Markov operators in our sense are often called *stochastic operators* in the literature on positive operators on AL and AM spaces.

This equilibrium concept is entirely standard.[12] The real number $\nu^*(B)$ gives the probability of being in Borel set B in this period, assuming that the current distribution is ν^*. The value $P\nu^*(B)$ gives the probability of being in B next period, given that the current distribution is ν^*. The equality of $\nu^*(B)$ and $P\nu^*(B)$ for each $B \in \mathcal{B}$ therefore implies that the probability law ν^*, once reached, is the law that governs the economy in the next and indeed all subsequent periods.

Note, however, that stability is defined here in terms of the norm topology. Existing techniques typically obtain only weak or weak-star stability.[13]

2.3.2 The L_1 Method

The framework introduced so far essentially follows Brock and Majumdar [9], Futia [21], Stokey et al. [58] and other previous work in economics. However, in this section we diverge slightly, approaching Markov chains generated by (2.4) using the L_1 method [29]; stochastic processes are studied by analyzing evolution of density functions which represent the marginal distributions of current and future state variables. The advantage is that we can exploit a very useful technique for studying Markov chains in L_1 due to Lasota [38].

Embedding the Markov problem in the function space $L_1(\mu)$ requires that

[12] See, for example, Brock and Mirman [10, p. 492], Futia [21, p. 377] or Stokey, Lucas and Prescott [58, pp. 317–8].

[13] Using coarser topologies is not a free lunch. After all, in every infinite dimensional normed space U, there exists a sequence of points all with norm one such that the sequence converges to the zero element in the weak topology induced by the norm dual of U.

the set of transition probabilities given in (2.8) can be represented by density functions. For the moment, let us simply assume that this is the case:

Assumption 2.1. For all $x \in X$, the distribution $B \mapsto \mathbf{N}(x, B)$ is absolutely continuous with respect to μ.

In the chapters that follow this assumption will be verified for different conditions on T and Ψ.

Denote the density that represents $B \mapsto \mathbf{N}(x, B)$ by $y \mapsto p(x, y)$. Heuristically, the number $p(x, y)dy$ is the probability of traveling from state x to state y in one step. In this paper, p is called the *density kernel* corresponding to (T, Ψ).

Let Assumption 2.1 hold, implying the existence of p. Using p, the Markov operator P corresponding to (T, Ψ) can now be reinterpreted as a linear self-mapping on the function space $L_1(\mu)$. Specifically, if $h \in L_1(\mu)$, then

$$Ph(y) = \int p(x, y)h(x)dx. \tag{2.11}$$

The two definitions (2.10) and (2.11) of P are equivalent for the absolutely continuous measures $\mathcal{M}_\mu \subset \mathcal{M}$ when these measures and their RN derivatives in $L_1(\mu)$ are identified. That is, if $h \in L_1(\mu)$ is the RN derivative of $\lambda \in \mathcal{M}_\mu$, then Ph defined by (2.11) is the RN derivative of $P\lambda$ defined by (2.10). To see this, pick any $B \in \mathcal{B}$. An application of Fubini's theorem gives

$$\begin{aligned}\int_B Ph(y)dy &= \int_B \int p(x,y)h(x)dxdy \\ &= \int \int_B p(x,y)dyh(x)dx \\ &= \int \mathbf{N}(x,B)h(x)dx \\ &= \int \mathbf{N}(x,B)\lambda(dx) \\ &= P\lambda(B).\end{aligned}$$

Formally, the semidynamical systems defined by $(\mathcal{M}_\mu, P)$, where $\mathcal{M}_\mu$ is the absolutely continuous measures and P is the Markov operator on measures; and $(L_1(\mu), P)$, where P is the Markov operator on functions; are topologically conjugate, in that they commute with the homeomorphism defined by Radon-Nikodým differentiation. Topologically conjugate dynamical systems have identical dynamic properties.

Note that $PD(\mu) \subset D(\mu)$, as can be shown directly using Fubini's theorem. Thus P is indeed a Markov operator (on L_1) in the sense of Definition 2.2, page 28. As before, if φ is the current marginal density for the state variable, then $P^t\varphi$ is that of the state t periods hence. Evolution of such a sequence of density functions is illustrated in Figure 2.3 for the case $X = \mathbb{R}$.

For systems evolving in $L_1(\mu)$, we redefine the equilibrium notion slightly.

Definition 2.4. Let (T, Ψ) be a stochastic dynamic economy on X satisfying Assumption 2.1. Let p be the associated density kernel, and let P be the Markov operator defined by (2.11). An *equilibrium* or *steady state* for (T, Ψ) is a density φ^* on X such that $P\varphi^* = \varphi^*$. An equilibrium φ^* is called *unique*

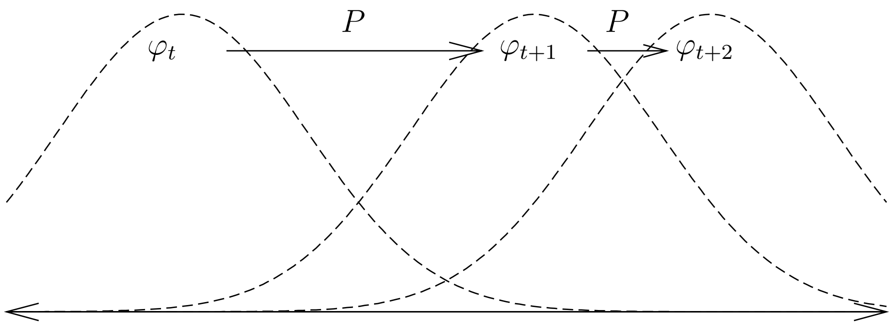

Figure 2.3: Evolution of densities on $\mathbb{R}$.

if there exists no other fixed point of P in the space $D(\mu)$, and *globally stable* if $P^t\varphi \to \varphi^*$ in the $L_1(\mu)$ metric as $t \to \infty$ for every $\varphi \in D(\mu)$.

2.4 General Fixed Point Results

Thus equilibria in Markovian models are defined to be fixed points in a particular semidynamical system (recall that the latter is a continuous self-mappings on topological space; c.f. Section 2.2.2, p. 20), where in the present case the self-mapping is the Markov operator P corresponding to the economy in question, and the domain is either the space of distributions $\mathcal{P}$, or the density functions $D(\mu)$. (The topology of the domain is that generated by the usual norm on the linear span of these spaces.)

This section considers some fixed point and stability results for arbitrary semidynamical systems. The key notions we will use are Lagrange stability and strong contractiveness.

It is shown that a semidynamical system which is Lagrange stable has at least one equilibrium, and that a semidynamical system which is strongly contractive has at most one equilibrium. Moreover, for a semidynamical system with both properties, the unique equilibrium is globally asymptotically stable.

While the ideas in this section draw heavily on Lasota [38], the approach and the proofs are presented here for the first time.

2.4.1 Lagrange Stability

Lagrange stability has been used extensively in the study of nonlinear differential equations and iterated function systems. Lagrange's original stability work was on the N-body problem of planetary motion. He showed that a first-order approximation of the system does not grow without bounds. The concept of Lagrange stability retains this meaning.

Recall that a subset A of topological space U is called *precompact* if A has compact closure.

Definition 2.5. Let U be a topological space and let T be a continuous self-mapping on U. Semidynamical system (U, T) is called *Lagrange stable* if the trajectory of x (i.e., the set of points $T^n x$, $n \in \mathbb{N}$) is precompact for every $x \in U$.

Remark 2.1. In finite dimensional space, precompactness is equivalent to boundedness by the Heine-Borel theorem. Thus for such a space Lagrange stability corresponds to the idea that none of the possible trajectories for the state variables grow without bounds.

A fixed point result for Lagrange stable systems is now stated. An alternative proof based on spectral decomposition can be found in Lasota and Mackey [40, Proposition 5.4.1], although the notation and formulation is somewhat different. Here a new proof is offered based on an infinite dimensional Brouwer (i.e., Schauder) fixed point theorem.[14]

Theorem 2.1. *Let V be a Banach space, and let U be a nonempty, closed and convex subset of V. Let T be a linear and continuous self-mapping on V such that $TU \subset U$. If (U, T) is Lagrange stable, then T has a fixed point in U.*

Proof. Take any $x \in U$. Define $\gamma(x)$ to be the set of all points $T^n x$, $n \in \mathbb{N}$; let $\hat{\gamma}(x)$ be its convex hull; and let $\mathrm{cl}(\hat{\gamma}(x))$ be the closure of the latter. Since the convex hull of a precompact set in V is again precompact [1, Theorem 5.20], it follows that $\hat{\gamma}(x)$ is precompact. Since the closure of a precompact set is compact, $\mathrm{cl}(\hat{\gamma}(x))$ must be compact. Using the linearity of T, if $a \in \hat{\gamma}(x)$, then evidently Ta is again in $\hat{\gamma}(x)$, or $T\hat{\gamma}(x) \subset \hat{\gamma}(x)$. But then $T\mathrm{cl}(\hat{\gamma}(x)) \subset \mathrm{cl}(\hat{\gamma}(x))$. The reason is that if A is any set with $TA \subset A$ and A' is the closure of A, then $TA' \subset A'$, because $a' \in A'$ implies the existence of a

[14]Schauder: Every self-mapping invariant on a nonempty compact convex subset of a locally convex Hausdorff space has at least one fixed point in that set.

sequence $(a_n) \subset A$, $a_n \to a'$, whence $Ta' = T \lim a_n = \lim Ta_n$, which, as the limit of a sequence in A, must again be in A'. Hence $TA' \subset A'$.

Thus T is invariant on nonempty convex compact set $\mathrm{cl}(\hat{\gamma}(x))$. But then T has a fixed point in $\mathrm{cl}(\hat{\gamma}(x))$ [32, Theorem 4.3.10]. Finally, since $\mathrm{cl}(\hat{\gamma}(x)) \subset U$ by the assumption that U is closed and convex, the fixed point must also be in U. $\square$

Remark 2.2. Note that Lagrange stability is quite a bit stronger than we actually require in the proof. Lagrange stability has been used here because of its relationship with subsequent results.

2.4.2 Strongly Contractive Systems

Next strong contractiveness and its relationship to Lagrange stability is discussed.

In many fields of economics, Banach's contraction principle is used to locate equilibria and solve dynamic programs.[15] Let (U, ϱ) be a metric space and let $T \colon U \to U$. The map T is called a *Banach contraction* if there exists an $\alpha < 1$ such that

$$\varrho(Tx, Ty) \leq \alpha \varrho(x, y), \quad \forall x, y \in U. \tag{2.12}$$

Banach's contraction principle is equivalent to the statement that if U is complete and T satisfies (2.12), then semidynamical system (U, T) is asymptotically stable [32, Theorem 4.1.1].

[15] See, for example, Stokey et al. [58, Lemma 11.11 and Section 17.2].

Unfortunately, for the semidynamical systems generated by stochastic growth models, in general (2.12) either does not hold or is difficult to verify. In contrast, the slightly weaker condition (2.14) below will prove both useful and easy to verify in applications.

Definition 2.6. Let topological space U be metrizable under distance ϱ, and let T be a continuous self-mapping on U. Semidynamical system (U, T) is called *contractive* (or *nonexpansive*) if

$$\varrho(Tx, Ty) \leq \varrho(x, y), \quad \forall x, y \in U. \tag{2.13}$$

The system is called *strongly contractive* if, in addition,

$$\varrho(Tx, Ty) < \varrho(x, y), \quad \forall x, y \in U, \ x \neq y. \tag{2.14}$$

Evidently (2.12) $\Longrightarrow$ (2.14) $\Longrightarrow$ (2.13). Like contractiveness in the sense of Banach, strong contractiveness implies uniqueness of equilibrium. (Suppose otherwise. In particular, let distinct points x and y be stationary under T. Then both $d(x, y) = d(Tx, Ty)$ and $d(Tx, Ty) < d(x, y)$. Contradiction.) However, strong contractiveness does not guarantee existence.[16] Nevertheless, existence and stability can be obtained if strong contractiveness is supplemented by compactness of U:

Lemma 2.1. *Let (U, T) be a semidynamical system, where U is a metrizable space. If (U, T) is strongly contractive and U is compact, then (U, T) is globally asymptotically stable.*

[16] For example, consider $U = \mathbb{R}_+$, $T \colon x \mapsto x + e^{-x}$.

Remark 2.3. Strictness of the inequality in (2.14) is necessary for both uniqueness and existence. For example, existence fails if U is the boundary of the unit sphere in $\mathbb{R}^2$, and $Tx = -x$.

Proof. This result is already known [32, Theorem 4.1.6, Corollary 1], but we include a proof for completeness. The stability part of the proof may be new.

Uniqueness has already been proved above. Consider the problem of existence. Define $r \colon U \to \mathbb{R}$ by $r(x) = \varrho(Tx, x)$. Evidently r is continuous. Since U is compact, r has a minimizer x^*. But then $Tx^* = x^*$, because otherwise Tx^* minimizes r on U.

Regarding stability, pick any $x \in U$, and define $\alpha_n = \varrho(T^n x, x^*)$. Since (α_n) is monotone decreasing and nonnegative it has a limit α. If $\alpha = 0$ then we are done. Suppose otherwise. By compactness, $(T^n x)$ has a convergent subsequence $T^{n_k} x \to \bar{x} \in U$. Evidently $\varrho(\bar{x}, x^*) = \alpha > 0$, so $\bar{x}$ and x^* are distinct. But then

$$\begin{aligned}\varrho(T\bar{x}, Tx^*) &= \varrho(T \lim_k T^{n_k} x, x^*) \\ &= \lim_k \varrho(T T^{n_k} x, x^*) \\ &= \lim_k \alpha_{n_k+1} = \alpha,\end{aligned}$$

which contradicts (2.14). This argument proves convergence to the fixed point. □

We have proved that contractiveness of the operator and compactness of the space together imply existence, uniqueness and global stability of equilibrium. In the problems generated by Markov operators, however, compactness of the underlying space will not in general be satisfied. An alternative (weaker) criterion is Lagrange stability of the operator. In particular,

Theorem 2.2. *Let X be a metrizable space, let U be a nonempty closed subset of X and let $T \colon X \to X$ be a continuous function invariant on U. If (U, T) is both Lagrange stable and strongly contractive, then it is globally asymptotically stable.*

Proof. Fix $x \in U$. Define $\Gamma(x)$ to be the closure of $\{T^n x : n \in \mathbb{N}\}$. Since (U, T) is Lagrange stable, $\Gamma(x)$ is a compact subset of X. Moreover, $T\Gamma(x) \subset \Gamma(x)$. Therefore $(\Gamma(x), T)$ is itself a strongly contractive semidynamical system on a compact set, and, by Lemma 2.1, has a unique fixed point $x^* \in \Gamma(x)$ with $T^n x \to x^*$. The point x^* is in U because U is closed and hence $\Gamma(x) \subset U$. Moreover, (U, T) has at most one fixed point by strong contractiveness. Therefore x^* does not depend on x. The result follows. □

2.5 Applications to Markov Operators

The objective of this section is to identify when Markov operators generated by stochastic dynamic economies might satisfy the conditions of the above fixed point theorems. These insights are entirely due to Lasota [38].

2.5.1 Contractive Markov Operators

Regarding contractiveness of Markov operators,

Lemma 2.2. *If P is a Markov operator on $L_1(\mu)$, then $(L_1(\mu), P)$ is contractive in the sense of* (2.13)*.*

Proof. Fix $f \in L_1(\mu)$. Define $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. By linearity and positivity,

$$|Pf(x)| = |Pf^+(x) - Pf^-(x)| \leq Pf^+(x) + Pf^-(x) = P|f(x)|.$$

Integration obtains

$$\|Pf\| = \int |Pf| d\mu \leq \int P|f| d\mu = \|f\|.$$

An application of linearity yields (2.13). □

An important sufficient condition for *strong* contractiveness of Markov operators on $D(\mu)$—which has been emphasized by A. Lasota—is as follows.

Lemma 2.3. *Let a perturbed dynamical system (T, Ψ) satisfying Assumption 2.1 be given, let p be the associated density kernel, and let $P\colon L_1(\mu) \to L_1(\mu)$ be the Markov operator defined from p by* (2.11)*. If $p > 0$ on $X \times X$, then P is strongly contractive on $D(\mu)$ with respect to the metric induced by the L_1 norm.*

Proof. Pick any two densities $\varphi \neq \varphi'$. Evidently the function $\varphi - \varphi'$ is both strictly positive on a set of positive measure and strictly negative on a set of positive measure. Pick any $y \in X$. Since $p(x, y) > 0$, it follows that $x \mapsto p(x, y)[\varphi(x) - \varphi'(x)]$ is also strictly positive on a set of positive measure and strictly negative on a set of positive measure. Therefore, by the strict triangle inequality,

$$
\begin{aligned}
\|P\varphi - P\varphi'\| &= \|P(\varphi - \varphi')\| \\
&= \int \left| \int p(x,y)[\varphi(x) - \varphi'(x)]dx \right| dy \\
&< \int \int \big| \, p(x,y)[\varphi(x) - \varphi'(x)] \, \big| \, dx \, dy \\
&= \int \int p(x,y)|\varphi(x) - \varphi'(x)|dx \, dy \\
&= \int \int p(x,y)dy|\varphi(x) - \varphi'(x)|dx \\
&= \|\varphi - \varphi'\|,
\end{aligned}
$$

which proves (2.14). □

2.5.2 Lagrange Stability of Markov Operators

The condition for strong contractiveness of P on $D(\mu)$ is therefore relatively straightforward. The other half of our primary stability condition, Theorem 2.2, is Lagrange stability. Here we discuss some strategies for establishing this property.

Lasota [38, Theorem 4.1] has made the important insight that in the case of *integral* Markov operators such as (2.11), it is sufficient to prove that $\{P^t\varphi : t \geq 0\}$ is *weakly* precompact for every $\varphi \in D(\mu)$. The reason is that integral Markov operators map weakly precompact subsets of $L_1(\mu)$ into strongly precompact subsets.[17] Therefore, if $\{P^t\varphi : t \geq 0\}$ is weakly precompact, then $\{P^t\varphi : t \geq 1\}$ is strongly precompact. But then $\{P^t\varphi : t \geq 0\}$ is also strongly precompact.

[17] This result appears to be due to A. Krasnosielski.

In fact Lasota [38, Proposition 3.4] has used a Cantor diagonal argument to show that weak precompactness of $\{P^t \varphi : t \geq 0\}$ need only be established for a collection of φ such that the norm closure of the collection contains $D(\mu)$.

Therefore,

Proposition 2.1 (Lasota). *If $P \colon D(\mu) \to D(\mu)$ is an integral Markov operator such as* (2.11)*, then the semidynamical system $(D(\mu), P)$ is Lagrange stable if and only if the set of functions $\{P^t \varphi : t \geq 0\}$ is weakly precompact for every $\varphi \in \mathcal{D}$, $\mathcal{D}$ a norm-dense subset of $D(\mu)$.*

To sum up our discussion so far,

Proposition 2.2. *Let (T, Ψ) be a stochastic dynamic economy evolving on state space X. Suppose that Assumption 2.1 holds, and let p be the associated density kernel. Suppose further that $p > 0$ on $X \times X$. If there exists a set $\mathcal{D} \subset D(\mu)$ such that $\mathcal{D}$ is norm-dense in $D(\mu)$ and $\{P^t \varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$, then (T, Ψ) has a unique and globally stable equilibrium in the sense of Definition 2.4.*

Proof. The proof is immediate from Theorem 2.2, Lemma 2.3 and Proposition 2.1. $\square$

2.5.3 Weak Precompactness

Before closing, we briefly recall a well-known characterization of weak precompactness in L_1 due to Dunford and Pettis [15, Theorem 3.2.1]. Given any

σ-finite measure space $(X, \mathcal{B}, \mu)$, a collection of functions $\{f_\alpha\}_{\alpha\in\Lambda}$ in $L_1(\mu)$ is a weakly precompact subset of that space if and only if it is norm bounded and the following two conditions hold:

(i) $\forall\varepsilon > 0$, $\exists\,\delta > 0$ such that if $A \in \mathcal{B}$ and $\mu(A) < \delta$, then

$$\int_A |f_\alpha| d\mu < \varepsilon, \quad \forall\alpha \in \Lambda; \text{ and}$$

(ii) $\forall\varepsilon > 0$, $\exists\, G \in \mathcal{B}$ such that $\mu(G) < \infty$ and

$$\int_{X\setminus G} |f_\alpha| d\mu < \varepsilon, \quad \forall\alpha \in \Lambda.$$

Evidently it is sufficient to verify that these conditions are satisfied for all but a finite number of the collection $\{f_\alpha\}$.

Chapter 3

Optimal Growth with Unbounded Shock

3.1 Introduction

This first essay studies equilibria in the stochastic optimal growth economy of Brock and Mirman [10], but without the assumption that the shock which perturbs production is realized within a bounded interval. It provides sufficient conditions for existence, uniqueness and stability of equilibria in terms of the primitives of the one-sector model, namely the utility function u, the per capita production function f and the distribution ψ of the disturbance term ε.

The original work of Brock and Mirman extends the deterministic optimal growth problem of Ramsey [50], Cass [11], Koopmans [34] and others to a stochastic setting. With regard to equilibria, they show that the existence, uniqueness and stability results of the deterministic case are also realized in a stochastic model under similar assumptions on preferences and production technology. In their analysis, the productivity shock is restricted to a bounded interval of the real line.[1]

The problem of characterizing equilibria and long-run behavior in Brock-Mirman economies with bounded shock has subsequently been studied by Mirman [45], Mirman and Zilcha [46], Brock and Majumdar [9], Razin and Yahav [51], Donaldson and Mehra [13], Majumdar and Zilcha [43], Stokey et al. [58], Hopenhayn and Prescott [28] and Amir [4]. The analogous problem for the overlapping generations model with bounded shock has been studied by Laitner [36] and Wang [62]. The related question of ergodicity in moments for the Solow-Swan model with a shock that is unbounded above but cannot be arbitrarily small is investigated in Binder and Pesaran [8]. Evstigneev and Flåm [17] and Amir and Evstigneev [5] investigate the asymptotic distributions of aggregate rewards accumulated along equilibrium and optimal paths. More general studies of stochastic equilibria in economics include Futia [21] and Duffie et al. [14].

Stochastic growth with unbounded shock is treated in Mirman [44], who provides an existence result and proves that the equilibrium measure is not concentrated at zero. His treatment leaves room for further analysis, however, as his sufficient conditions pertain to a class of consumption policies that may or may not be optimal [44, A1–A3, p. 275]. In other words, the savings

[1]Such a shock is said to have compact support. For simplicity these shocks are referred to as “bounded”. Shocks where no restrictions are placed on the support are called “unbounded”.

rate is exogenously given, and the conditions are not stated in terms of the primitives u, f and ψ. Further, the problems of uniqueness and stability are not treated. In the present essay, conditions for existence, uniqueness and stability are obtained in terms of the triple (u, f, ψ) and the restrictions imposed by optimizing behavior.

The mathematical techniques used in the essay are based on recent innovations in the theory of perturbed dynamical systems. The two key concepts are Lagrange stability and strong contractiveness.

In addition to identifying and characterizing equilibria in Brock-Mirman economies with unbounded shock, the essay also makes the following contributions. First, the L_1 approach is introduced to stochastic growth theory. Second, the notions of strong contractiveness and Lagrange stability are developed in the context of stochastic optimal growth.

The structure of the essay is as follows. Section 3.2 formulates the stochastic optimal growth problem. Section 3.3 states the main result. The proof is then developed over Sections 3.4–3.5.

3.2 Formulation of the Problem

This section contains a formulation of the stochastic optimal growth problem studied by Brock and Mirman [10]. The symbols $\mathbb{R}_+$ and $\mathbb{R}_{++}$ denote the nonnegative and positive reals respectively. Given any topological space X, $\mathcal{B}(X)$ denotes the Borel sets of X. All sets of real numbers introduced in the essay are assumed to be Borel sets, and all real functions are Borel functions.

Lebesgue measure is denoted by μ. Unless otherwise stated, integration is with respect to μ.

The accumulation problem evolves as follows. At the start of period t the (representative) agent receives income x_t. In response a level of consumption $c_t \leq x_t$ is chosen, yielding current utility $u(c_t)$. The remainder is invested in production, returning in the following period output $x_{t+1} = f(x_t - c_t)\varepsilon_t$. Here f is the production function and ε is a nonnegative random variable.[2] The process then repeats.

3.2.1 Assumptions

The functions u and f satisfy the usual assumptions.

Assumption 3.1. The production function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$ is zero at zero, strictly increasing, strictly concave, differentiable and satisfies the Inada conditions $\lim_{x\downarrow 0} f'(x) = \infty$ and $\lim_{x\uparrow\infty} f'(x) = 0$.

Assumption 3.2. The utility function $u\colon \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, differentiable and satisfies $\lim_{x\downarrow 0} u'(x) = \infty$.

The shock is permitted to be unbounded.

Assumption 3.3. The shocks to production are uncorrelated and identically distributed. The distribution of ε is represented by density ψ. The shock

[2] Following Stokey et al. [58] and Hopenhayn and Prescott [28], it is assumed that the disturbance term ε is multiplicative. Brock and Mirman use the more general formulation $x_{t+1} = f(x_t - c_t, \varepsilon_t)$. See Amir [4] for an even more general technology.

has finite mean $\mathbb{E}(\varepsilon)$. In addition, ε satisfies $\mathbb{E}(1/\varepsilon) < 1$. The shock is less than one with positive probability, i.e. $\int_0^1 \psi(x)dx \neq 0$.

3.2.2 Technology

The conditional density for next-period output given income x and consumption c is, by a change of variable argument,

$$y \mapsto \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}. \tag{3.1}$$

Given that $f(0) = 0$, (3.1) is not defined when consumption is equal to income. In this case (when $c = x$), next-period income is zero with probability one. Such a probability cannot be represented by a density. Consequently, the fully specified technology associating savings $x - c$ to next-period income will be defined by probability $B \mapsto \mathbf{Q}(x, c; B)$, where

$$\mathbf{Q}(x, c; B) = \int_B \psi\left(\frac{y}{f(x-c)}\right)\frac{1}{f(x-c)}dy.$$

when $c < x$, and by the probability concentrated at zero when $c = x$. Thus $\mathbf{Q}(x, c; B)$ is the probability that next-period output is in B given that current income is x and consumption is $c \in [0, x]$.

3.2.3 The Optimal Policy

Future utility is discounted geometrically at rate $\beta \in (0, 1)$. The agent selects a sequence (c_t) to solve

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{3.2}$$

subject to the feasibility constraint $0 \leq c_{t+1} \leq f(x_t - c_t)\varepsilon_t$.

The meaning of the expectations operator in (3.2) is not immediately clear. A more formal statement of the problem is that the agent seeks a control policy $g \colon \mathbb{R}_+ \ni x_t \mapsto c_t \in \mathbb{R}_+$ that is feasible (i.e., $g(x) \in [0, x]$) and maximizes $v(x, g)$, where

$$v(x, g) = \mathbb{E}_x^g \left[\sum_{t=0}^{\infty} \beta^t u(g(x_t)) \right].$$

Here $\mathbb{E}_x^g$ signals integration with respect to the (well-defined and unique) Markovian distribution over infinite-dimensional sequence space $\mathbb{R}_+^{\mathbb{N}}$ generated by Markov transition kernel $\mathbf{Q}(x, g(x); dy)$.[3]

The value function V for the problem is defined at x as the supremum of $v(x, g)$ over the set of all feasible policies. A feasible policy g^* is called *optimal* if $v(x, g^*) = V(x)$ for all x.

The following results are well-known.

Theorem 3.1. *Let u, f and ψ satisfy Assumptions 3.1–3.3. The following results hold.*

1. *The value function V is finite and satisfies the Bellman equation*

$$V(x) = \max_{0 \leq c \leq x} \left\{ u(c) + \beta \int V(y) \mathbf{Q}(x, c; dy) \right\}.$$

2. *There exists a unique optimal policy g and*

$$V(x) = u(g(x)) + \beta \int V(y) \mathbf{Q}(x, g(x); dy).$$

[3]See, for example, Hernández-Lerma and Lasserre [26].

3. *The value function is nondecreasing, concave and differentiable with*

$$V'(x) = u'(g(x)).$$

4. *If g is an optimal policy, then $0 < g(x) < x$, $\forall x > 0$, and both $x \mapsto g(x)$ and $x \mapsto x - g(x)$ are nondecreasing (savings and consumption both increase with income).*

Proof. See, for example, Mirman and Zilcha [46, p. 331–2]. (For a formal discussion of Markov control programs with unbounded reward see Hernández-Lerma and Lasserre [27, Chapter 8].) Here parts 1–3 of the theorem $\Longrightarrow$ part 4. □

Substitution of the optimal control into the production relation yields the closed-loop law of motion

$$x_{t+1} = f(x_t - g(x_t))\varepsilon_t. \tag{3.3}$$

3.3 Statement of Results

It is now possible to state our main result, which gives sufficient conditions for existence, uniqueness and stability of equilibria in the stochastic growth model of the previous section. It shows that the original results of Brock and Mirman also hold for many of the standard (unbounded) shocks used in mathematical statistics.

Theorem 3.2. *Let u, f and ψ satisfy Assumptions 3.1–3.3. The following statements are true.*

1. *The economy (u, f, ψ) has at least one (nonzero) equilibrium.*

2. *If, in addition, ψ is everywhere positive, then the equilibrium is unique and globally stable.*

The proof is developed in stages through the remaining sections. The approach is to represent the economy (u, f, ψ) as a semidynamical system and then apply the concepts of Lagrange stability and strong contractiveness.

In Section 3.4 it is shown that (u, f, ψ) can be represented as a semidynamical system formed by a Markov operator on the space of density functions. If it can be established under Assumptions 3.1–3.3 that this semidynamical system generated by (u, f, ψ) is Lagrange stable, then Theorem 2.1, page 34, can be used to demonstrate the existence of at least one equilibrium. If, in addition, it can be shown that positivity of ψ in part 2 of Theorem 3.2 implies strong contractiveness, then by Theorem 2.2, page 38, the system is also asymptotically stable, which is to say that there exists a unique and globally stable equilibrium.[4] Lagrange stability and strong contractiveness are established in Section 3.5, completing the proof of the Theorem 3.2.

The proof of Lagrange stability (Proposition 3.1, p. 57) constitutes the main technical contribution of the essay. As expected, the Inada conditions and the concavity of the program are crucial to the proof.

[4]See Definition 2.4, page 31.

3.3.1 Examples

Let f and u satisfy Assumptions 3.1 and 3.2 respectively, and let the density ψ of ε be lognormal. In other words, $\ln \varepsilon$ is normally distributed with mean m and variance σ^2, $\sigma > 0$. Since $\mathbb{E}(1/\varepsilon) = \exp(\sigma^2/2 - m)$, $\mathbb{E}(1/\varepsilon) < 1$ if $\sigma^2/2 < m$. In this case, all of the components of Assumption 3.3 are also satisfied, and (u, f, ψ) has at least one equilibrium. In addition, the density function is everywhere positive. It follows from part 2 of the theorem that the equilibrium is unique and globally stable.

In fact the same result holds for *any* lognormal shock. To see this, let m and σ be arbitrary, $\sigma > 0$, and let θ be a constant strictly larger than $\mathbb{E}(1/\varepsilon)$. If $\varepsilon^* = \theta\varepsilon$, $f^* = (1/\theta)f$ and ψ^* is the distribution of ε^*, then (u, f^*, ψ^*) satisfies Assumptions 3.1–3.3 and all of the conditions of the theorem. Hence (u, f^*, ψ^*) has a unique, globally stable equilibrium. But

$$f^*(x - c)\varepsilon^* = \frac{1}{\theta} f(x - c)\theta\varepsilon = f(x - c)\varepsilon,$$

so (u, f, ψ) and (u, f^*, ψ^*) are identical.[5] It follows that (u, f, ψ) also has a unique, globally stable equilibrium.

3.3.2 Remarks

The restriction

$$\mathbb{E}(1/\varepsilon) = \int_0^\infty \frac{1}{x}\psi(x)dx < 1 \tag{3.4}$$

[5] More formally, both economies have the same density kernel.

used in the theorem has a simple interpretation. Previous work has assumed that ε is realized in a compact interval $[a, b]$, $0 < a \leq b < \infty$. Here, in contrast, the shock may be arbitrarily large or arbitrarily close to zero. Equation (3.4) implies that ε is "unlikely" to be very close to zero, or, in other words, that the left-hand tail of the density ψ is relatively small. To see this, define for nonnegative summable function V and for density h on $\mathbb{R}_{++}$ the (possibly infinite) number

$$E(V|h) = \int_{\mathbb{R}_{++}} V(x)h(x)dx, \tag{3.5}$$

as well as the set $G_a = \{x \in \mathbb{R}_{++} : V(x) < a\}$. Evidently,

$$E(V|h) \geq \int_{\mathbb{R}_{++}\setminus G_a} V(x)h(x)dx,$$

implying

$$\int_{\mathbb{R}_{++}\setminus G_a} h(x)dx \leq \frac{E(V|h)}{a}. \tag{3.6}$$

This is in fact a version of Chebychev's inequality. Substituting $I^{-1}\colon x \mapsto x^{-1}$ for V, ψ for h and $1/r$ for a gives

$$\int_0^r \psi(x)dx \leq r\mathbb{E}(1/\varepsilon).$$

Thus (3.4) is a restriction on the left-hand tail of ψ.

It has also been assumed that $\mathbb{E}(\varepsilon)$ is finite. This is a restriction on the right-hand tail. To see this, substitute $I\colon x \mapsto x$ for V and ψ for h to obtain

$$\int_a^\infty \psi(x)dx \leq \frac{\mathbb{E}(\varepsilon)}{a}. \tag{3.7}$$

These restrictions on the tails of ψ can be thought of as a generalization of the assumption that ψ is zero below a and above b made in previous studies.

(As a caveat to the claim that the restrictions on ψ are a generalization of boundedness, recall that in this paper—in contrast to the majority of previous work—the shock must be non-degenerate and representable by a density function.)

The assumption on positivity of ψ in part 2 of the theorem is akin to the "communication" assumptions used in traditional Markov chain theory [52, Chapter 8].

3.4 Semidynamical Systems

In this section it is shown that (u, f, ψ) can be interpreted as a semidynamical system defined by a Markov operator on the space of density functions.

Following the notation in Chapter 2, let $L_1(\mu)$ be the Banach lattice of μ-integrable real functions on $\mathbb{R}_{++}$, and let $D(\mu)$ be the set of $h \in L_1(\mu)$ such that $h \geq 0$ and $\|h\| = 1$.

The density kernel, Markov operator and semidynamical system associated with the Brock-Mirman economy (u, f, ψ) are derived from the law of motion (3.3).

By a change of variable argument, the conditional density for next-period income given that current income equals x is

$$y \mapsto \psi\left(\frac{y}{f(x - g(x))}\right)\frac{1}{f(x - g(x))}. \tag{3.8}$$

As a function of both x and y, (3.8) defines a density kernel for measure space $(\mathbb{R}_{++}, \mathcal{B}(\mathbb{R}_{++}), \mu)$ in the sense of Chapter 2. Denote by Q the Markov operator associated with (3.8) by (2.11). The semidynamical system for the Brock-Mirman process is then $(D(\mu), Q)$. If initial income x_0 is distributed according to φ_0, then time t income is distributed according to $Q^t\varphi_0$.

A plot of (3.8) is shown in Figure 3.1 for the parameterization $f\colon x \mapsto x^{1/2}$, $u\colon x \mapsto \ln x$, ε lognormal. The origin is the corner of the graph furthest from the viewer. For each x_t, a density function runs parallel to the x_{t+1} axis. The density governs the likelihood that income per head takes values along that axis, given that the current state is x_t.[6]

3.5 Proof of the Main Theorem

The proof of Theorem 3.2 proceeds as follows. After some preliminary results, Lagrange stability of the semidynamical system associated with (u, f, ψ) is established. Next, strong contractiveness of the economy is established using the additional hypothesis of positivity of ψ. The proof is then completed in Section 3.5.2.

3.5.1 Proof of Lagrange Stability

This first lemma is required in the proof of Lagrange stability of the semidynamical system $(D(\mu), Q)$ associated with (u, f, ψ).

[6]For a kernel estimated nonparametrically from actual growth data see Quah [48, Figures 5 and 6].

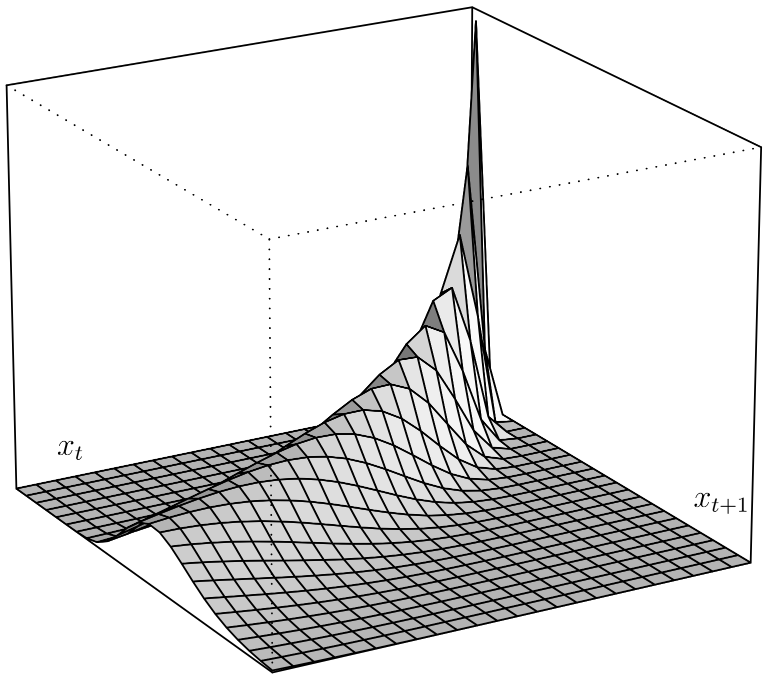

Figure 3.1: Density kernel (3.8).

Lemma 3.1. *Let (u, f, ψ) satisfy Assumptions 3.1–3.3. If g is an optimal policy, then there exists an $x_0 > 0$ such that*

$$f(x - g(x)) \geq x \text{ whenever } x \in (0, x_0].$$

Proof. The first order condition of Theorem 3.1, part (1) is

$$u'(g(x)) = \beta \int_0^\infty V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz.$$

Using the envelope relation of Theorem 3.1, part (3) obtains

$$\begin{aligned} V'(x) &= \beta \int_0^\infty V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz \\ &\geq \beta \int_0^1 V'(f(x - g(x))z) f'(x - g(x)) z\psi(z) dz \\ &\geq \beta \int_0^1 V'(f(x - g(x))) f'(x - g(x)) z\psi(z) dz, \end{aligned}$$

where the first inequality follows from the fact that V is nondecreasing and the second from the fact that V is concave. Thus,

$$V'(x) \geq V'(f(x - g(x))) f'(x - g(x)) M, \quad M = \beta \int_0^1 z\psi(z) dz.$$

The constant M is positive by Assumption 3.3. Assumption 3.1 and the monotonicity of $x \mapsto x - g(x)$ then imply the existence of an $x_0 > 0$ such that $f'(x - g(x))M \geq 1$ whenever $x \in (0, x_0]$. Therefore,

$$V'(x) \geq V'(f(x - g(x))) \text{ on } (0, x_0].$$

The result now follows from the concavity of V. □

The proof of the following proposition draws extensively on methods developed by Lasota [38] and Horbacz [30].

Proposition 3.1. *If (u, f, ψ) satisfies Assumptions 3.1–3.3, then the associated semidynamical system $(D(\mu), Q)$ is Lagrange stable.*

Proof. By Proposition 2.1 on page 41, it is sufficient to find a $\mathcal{D} \subset D(\mu)$ such that $\mathcal{D}$ is dense in $D(\mu)$ and the set $\{Q^n h\}$ is weakly precompact for every h in $\mathcal{D}$.

Let $\mathcal{D}$ be the collection of densities h that satisfy

$$\int_0^\infty xh(x)dx < \infty \text{ and } \int_0^\infty \frac{1}{x} h(x)dx < \infty. \tag{3.9}$$

We claim that $\mathcal{D}$ has the desired properties. To see that $\mathcal{D}$ is dense in the densities, fix $\varphi \in D(\mu)$ and define $h_k^0 = \mathbf{1}_{(1/k,k)}\varphi$. Since $\|h_k^0\| \uparrow 1$ by the monotone convergence theorem, it follows that for some $K \in \mathbb{N}$, $\|h_k^0\| > 0$ whenever $k \geq K$. For all such k define

$$h_k = \|h_k^0\|^{-1} h_k^0.$$

It can be established that h_k satisfies (3.9) for each k. In addition, h_k is a density by construction, and $h_k \to \varphi$ pointwise. But then $h_k \to \varphi$ in the L_1 norm by Scheffé's lemma [59, Proposition 4.5.14]. Thus $\mathcal{D}$ is dense in $D(\mu)$.

It remains to show that that if $h \in \mathcal{D}$ then $\{Q^n h\}_{n \geq 1}$ is weakly precompact. Fix arbitrary $h \in \mathcal{D}$. It is sufficient to establish precompactness of $\{Q^n h\}_{n \geq N}$ for some fixed $N \in \mathbb{N}$, because appending a finite number of elements to a (weakly) precompact set does not alter the property of (weak) precompactness. We now show that $\{Q^n h\}_{n \geq N}$ is weakly precompact for some $N \in \mathbb{N}$ by verifying the Dunford-Pettis conditions given in Section 2.5.3, page 41.

Boundedness of the collection is satisfied because $\|Q^n h\| = \|h\| = 1$ for all n by the positive isometry property of Markov operators (Definition 2.2). We now show that conditions (i) and (ii) of the Dunford-Pettis condition also hold (c.f. p. 41).

For notational simplicity define $q(x) = f(x - g(x))$. Use will be made of the fact that

$$\frac{1}{q(x)}\mathbb{E}(1/\varepsilon) \leq \gamma\frac{1}{x} + C \tag{3.10}$$

for all positive x, where γ and C are nonnegative constants, $\gamma < 1$.

To verify (3.10), recall that $\exists x_0 > 0$ such that $q(x) \geq x$ when $x \leq x_0$ by Lemma 3.1. Choose any γ such that $\mathbb{E}(1/\varepsilon) < \gamma < 1$. Then

$$\frac{1}{q(x)}\mathbb{E}(1/\varepsilon) \leq \gamma\frac{1}{x}, \quad \forall x \leq x_0. \tag{3.11}$$

Moreover, on $[x_0, \infty)$, monotonicity of f and $x \mapsto x - g(x)$ implies that $q(x) \geq q(x_0)$, or

$$\frac{1}{q(x)}\mathbb{E}(1/\varepsilon) \leq \frac{1}{q(x_0)}\mathbb{E}(1/\varepsilon) = C. \tag{3.12}$$

Together, (3.11) and (3.12) imply (3.10).

Let I^{-1} be the map $x \mapsto x^{-1}$. Applying in succession (3.5), (3.8), Fubini's theorem, a change of variable argument and (3.10),

$$\begin{aligned}
E(I^{-1}|Q^n h) &= \int_0^\infty \frac{1}{y} Q^n h(y) dy \\
&= \int_0^\infty \frac{1}{y} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} Q^{n-1} h(x) dx \right] dy \\
&= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right) \frac{1}{q(x)} \frac{1}{y} dy \right] Q^{n-1} h(x) dx \\
&= \int_0^\infty \frac{1}{q(x)} \mathbb{E}(1/\varepsilon) Q^{n-1} h(x) dx \\
&\leq \int_0^\infty [\gamma \frac{1}{x} + C] Q^{n-1} h(x) dx \\
&= \gamma E(I^{-1}|Q^{n-1} h) + C.
\end{aligned}$$

Repeating the argument n times,

$$E(I^{-1}|Q^n h) \leq \gamma^n E(I^{-1}|h) + \frac{C}{1-\gamma},$$

or, using finiteness of $E(I^{-1}|h)$,

$$E(I^{-1}|Q^n h) \leq 1 + \frac{C}{1-\gamma}$$

when $n \geq K$, K suitably large.

An application of the Chebychev argument (3.6) gives

$$\int_0^r Q^n h(x) dx \leq r E(I^{-1}|Q^n h)$$

for any positive r. Therefore,

$$\int_0^r Q^n h(x) dx \leq r \left(1 + \frac{C}{1-\gamma}\right), \quad n \geq K. \tag{3.13}$$

Now fix any $\varepsilon > 0$. According to the Dunford-Pettis condition part (i), we require a $\delta > 0$ and a $K \in \mathbb{N}$ such that $n \geq K$ implies

$$\int_A Q^n h(x) dx < \varepsilon$$

whenever $\mu(A) \leq \delta$. For this purpose, consider the decomposition

$$\int_A Q^n h(x)dx = \int_{A\cap(0,r)} Q^n h(x)dx + \int_{A\cap(r,\infty)} Q^n h(x)dx. \tag{3.14}$$

Using (3.13) gives

$$\int_{A\cap(0,r)} Q^n h(x)dx \leq \int_0^r Q^n h(x)dx < \frac{\varepsilon}{2}. \tag{3.15}$$

when $r > 0$ is chosen to be sufficiently small and $n \geq K$.

Take r as given and consider the second term in (3.14).

$$\begin{aligned}
\int_{A\cap(r,\infty)} Q^n h(x)dx &= \int_{A\cap(r,\infty)} \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}Q^{n-1}h(x)dx\right]dy \\
&= \int_0^\infty \left[\int_{A\cap(r,\infty)} \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}dy\right] Q^{n-1}h(x)dx. \\
&= \int_0^\infty \left[\int_{\frac{A\cap(r,\infty)}{q(x)}} \psi(z)dz\right] Q^{n-1}h(x)dx.
\end{aligned}$$

The term in brackets can be written as

$$G(x) = \int_{\frac{r}{q(x)}}^\infty \mathbf{1}_{\frac{A}{q(x)}}(z)\psi(z)dz.$$

By (3.7) it is possible to choose $\alpha > 0$ so small that

$$\int_{\frac{r}{q(\alpha)}}^\infty \psi(z)dz < \frac{\varepsilon}{2}.$$

Evidently,

$$G(x) < \frac{\varepsilon}{2} \text{ whenever } x \leq \alpha.$$

Now consider the case where $x > \alpha$. Select $\delta' > 0$ such that

$$\mu(B) < \delta' \Longrightarrow \int_B \psi(z)dz < \frac{\varepsilon}{2}.$$

Existence of such a δ' follows from absolute continuity of the integral measure with respect to μ [59, Ex. 2.8.15]. Define $\delta = q(\alpha)\delta'$. Then $x > \alpha$ and $\mu(A) < \delta$ implies

$$G(x) \leq \int_{\frac{A}{q(x)}} \psi(z)dz < \frac{\varepsilon}{2}$$

because $\mu(A/q(x)) = \mu(A)/q(x) \leq \mu(A)/q(\alpha) < \delta/q(\alpha) = \delta'$. Thus $\mu(A) < \delta$ implies $G(x) < \varepsilon/2$ for all x, and hence

$$\int_{A\cap(r,\infty)} Q^n h(x)dx < \frac{\varepsilon}{2}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16) gives

$$\int_A Q^n h(x)dx < \varepsilon$$

when $\mu(A) < \delta$ and $n \geq K$. Thus part (i) of the Dunford-Pettis condition holds for the collection $(Q^n h)_{n\geq K}$.

Next, part (ii) of the condition needs to be checked for the same h. Let I be the identity map on $\mathbb{R}_{++}$. We have

$$\begin{aligned}
E(I|Q^n h) &= \int_0^\infty yQ^n h(y)dy \\
&= \int_0^\infty y\left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}Q^{n-1}h(x)dx\right]dy \\
&= \int_0^\infty \left[\int_0^\infty \psi\left(\frac{y}{q(x)}\right)\frac{1}{q(x)}ydy\right]Q^{n-1}h(x)dx \\
&= \int_0^\infty \mathbb{E}(\varepsilon)q(x)Q^{n-1}h(x)dx \\
&\leq \int_0^\infty \mathbb{E}(\varepsilon)f(x)Q^{n-1}h(x)dx.
\end{aligned}$$

Since $\mathbb{E}(\varepsilon)$ is finite, it follows from the concavity and Inada conditions in Assumption 3.1 that $x \mapsto \mathbb{E}(\varepsilon)f(x)$ can be majorized on $\mathbb{R}_{++}$ by an affine function with slope less than one. In other words, there exist nonnegative constants a and b, $a < 1$, such that $\mathbb{E}(\varepsilon)f(x) \leq ax + b$, $\forall x > 0$. Therefore

$$\begin{aligned}\int_0^\infty \mathbb{E}(\varepsilon)f(x)Q^{n-1}h(x)dx &\leq \int_0^\infty [ax+b]Q^{n-1}h(x)dx \\ &= aE(I|Q^{n-1}h) + b.\end{aligned}$$

Repeating the argument n times,

$$E(I|Q^n h) \leq a^n E(I|h) + \frac{b}{1-a},$$

or, using finiteness of $E(I|h)$,

$$E(I|Q^n h) \leq 1 + \frac{b}{1-a}$$

when $n \geq M$, M suitably large.

By (3.6),

$$\int_r^\infty Q^n h(x)dx \leq \frac{E(I|Q^n h)}{r}$$

for any n and any positive r. Hence

$$\int_r^\infty Q^n h(x)dx \leq \frac{1}{r}\left(1 + \frac{b}{1-a}\right), \quad n \geq M,$$

and condition (ii) of the Dunford-Pettis condition holds for $\{Q^n h\}_{n \geq M}$.

Finally, define $N = \max(K, M)$. For such an N, $\{Q^n h\}_{n \geq N}$ satisfies both parts of the Dunford-Pettis condition, completing the proof of Lagrange stability. $\square$

Regarding strong contractiveness of the semidynamical system $(D(\mu), Q)$, the following statement is true.

Proposition 3.2. *If the density ψ of the shock ε is everywhere positive, then $(D(\mu), Q)$ is strongly contractive.*

Proof. The result follows immediately from implied positivity of the density kernel (3.8) and Lemma 2.3, page 39. □

3.5.2 Proof of Theorem 3.2

It is now possible to prove Theorem 3.2. Proposition 3.1 shows that if (u, f, ψ) satisfies Assumptions 3.1–3.3, then the associated semidynamical system $(D(\mu), Q)$ is Lagrange stable. Evidently $D(\mu)$ is a closed convex subset of $L_1(\mu)$. Moreover Q is both linear and continuous. Hence all the conditions of Theorem 2.1, page 34, are satisfied, implying the existence of an equilibrium density. Since the equilibrium is a density, probability is not concentrated at zero (i.e. it is a nonzero equilibrium). Regarding part 2 of Theorem 3.2, if, in addition, ψ is assumed to be everywhere positive, then $(D(\mu), Q)$ is also strongly contractive by Proposition 3.2. Existence, uniqueness and stability of equilibrium now follow from Theorem 2.2, page 38.

Chapter 4

Systems with Multiplicative Noise

4.1 Introduction

In this essay we focus on existence, uniqueness and stability of equilibrium in a specific class of models that arise naturally in economics. In particular, we assume that the shock ε in (2.4), page 22, is multiplicative, and that the state space for the endogenous variable x_t is the positive half-ray $\mathbb{R}_+ = [0, \infty)$. That is,

$$x_{t+1} = g(x_t)\varepsilon_t, \quad t = 0, 1, \ldots, \tag{4.1}$$

where $g\colon \mathbb{R}_+ \to \mathbb{R}_+$, and $\varepsilon_t \in \mathbb{R}_+$. The importance of these models within economics stems from inherently nonnegative state variables, such as prices or physical quantities. A key example is of course stochastic growth theory, which in turn provides foundations for real business cycle and other macroeconomic literature.

While (4.1) excludes a larger model architecture than previous studies of the equilibrium problem, it is demonstrated that the additional structure can be exploited to obtain results that have considerable generality within this class. Further, our approach leads naturally to sufficient conditions stated directly in terms of the primitives g and ε; such conditions are easy to verify in applications. Third, the temptation to compactify the state space is resisted, permitting incorporation of standard econometric shocks. Fourth, equilibria are realized as fixed points of a contractive linear operator, and are therefore amenable to approximation by numerical methods.

The stability of (4.1) has previously been studied in the mathematical literature. In particular, there exists a well-known set of sufficient conditions due to K. Horbacz [30, Theorem 1]. The results obtained here provide a general principle which yields the conditions of Horbacz as a special case.

The essay proceeds as follows. Section 4.2 states our results. Section 4.3 gives applications. Section 4.4 formalizes the problem as a preliminary to the proofs. The proofs are collected in Section 4.5.

4.2 Results

In this section we state new stability results for the economy on $\mathbb{R}_+$ defined by (4.1). The results pertain to the existence, uniqueness and stability of stochastic equilibria. Our definition of this property is the standard one:

existence of a unique "stationary" distribution on the state space to which the marginal distributions of the state variables (x_t) always converge as $t \to \infty$, regardless of the initial state. (Recall Definition 2.3, p. 28.) Formal definitions of the distribution space and topology of convergence are given in Section 4.4.

Our basic assumptions on the structure of the model (4.1) are as follows. First,

Assumption 4.1. The shocks (ε_t) are uncorrelated and identically distributed by density function ψ on $\mathbb{R}_+$.

Second,

Assumption 4.2. The map g is strictly positive almost everywhere on $\mathbb{R}_+$.[1]

In the remainder of the essay, the model (4.1) is represented by notation (g, ψ), where ψ is the density of the shock ε.

Our main condition uses the notion of a *Lyapunov function on* $\mathbb{R}_+$, which we define to be a continuous, nonnegative function V from $\mathbb{R}_+$ into $\mathbb{R}_+ \cup \{\infty\}$ such that $V(0) = \infty$, $V(x) < \infty$ for $x > 0$ and $\lim_{x\to\infty} V(x) = \infty$.

Condition 4.1. Corresponding to (g, ψ), there exists a Lyapunov function V on $\mathbb{R}_+$ and constants $\alpha, C \geq 0$, $\alpha < 1$, such that

$$\int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C, \quad \forall x \in \mathbb{R}_+.$$

[1] Thus we accommodate the possibility that g may be zero at a finite number of points.

The function V in Condition 4.1 is large at 0 and $+\infty$. The condition should be interpreted as a restriction on the probability that the state variable moves toward these limits without bound.

Condition 4.2. The density ψ is strictly positive almost everywhere on $\mathbb{R}_+$.[2]

Most "named" densities on $\mathbb{R}_+$ have this property, such as the lognormal, exponential, χ-squared, gamma, and Weibull densities.

Condition 4.3. For some $M < \infty$, ψ satisfies $\psi(z)z \leq M$, $\forall z \in \mathbb{R}_+$.

Condition 4.3 also holds for the lognormal, exponential, χ-squared, gamma and Weibull distributions. The condition is used here to bound the probability that ψ assigns to closed intervals in $\mathbb{R}_+ \setminus \{0\}$.

Theorem 4.1. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 4.1 and 4.2. If g and ψ also satisfy Conditions 4.1, 4.2 and 4.3, then (g, ψ) has a unique, globally stable equilibrium.*

Alternatively, suppose that

Condition 4.4. The map g is weakly monotone increasing on the nonempty interval $[0, r)$, and $g(x) \geq b > 0$ on $[r, \infty)$.

Theorem 4.2. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 4.1 and 4.2. If g and ψ also satisfy Conditions 4.1, 4.2 and 4.4, then (g, ψ) has a unique, globally stable equilibrium.*

[2] When this is the case, the same distribution for ε can be represented by a density which is positive *everywhere* on $\mathbb{R}_+$. Hence in the remainder of the paper we can assume without loss of generality that $\psi(z) > 0$, $\forall z \in \mathbb{R}_+$.

The proofs of Theorems 4.1 and 4.2 are given in Section 4.5.

Corollary 4.1. *Let (g, ψ) be an economy on $\mathbb{R}_+$ satisfying Assumptions 4.1 and 4.2. If g is weakly monotone increasing and, in addition, g and ψ together satisfy Conditions 4.1 and 4.2, then (g, ψ) has a unique, globally stable equilibrium.*

Proof. Evidently Condition 4.4 is also satisfied if Assumption 4.2 holds and g is weakly monotone increasing on $\mathbb{R}_+$. Theorem 4.2 then implies the stated result. □

4.3 Applications

We analyze the dynamics of two capital accumulation models using our methods. One is a standard overlapping generations model, while the other is of optimal growth with externality-driven increasing returns.

4.3.1 Overlapping Generations

In the deterministic case, dynamics of the overlapping generations model with productive capital were extensively studied by Galor and Ryder [22]. They establish convergence to a unique, nontrivial equilibrium under a strengthened Inada condition. Here the analysis is extended to the stochastic case, proving that analogous results hold under the same condition.[3]

[3] Previously the stochastic overlapping generations model was also analyzed by Wang [62].

The framework is as follows. Agents live for two periods, working in the first and living off savings in the second. Savings in the first period forms capital stock, which in the following period will be combined with the labor of a new generation of young agents for production under the technology $y_{t+1} = F(k_t, \ell_t)\varepsilon_t$, where y is income, k is capital and ℓ is labor input. For convenience we assume that labor is constant ($\ell_t = \ell$), and set $f(k) = F(k, \ell)$. Following Galor and Ryder [22, p. 362], we assume that $f \colon \mathbb{R}_+ \to \mathbb{R}_+$ has the usual properties $f(0) = 0$, $f \in C^2$, $f' > 0$, $f'' < 0$, and

$$\lim_{k \downarrow 0} f'(k) = \infty, \qquad \lim_{k \uparrow \infty} f'(k) = 0.$$

In addition, Galor and Ryder [22, Proposition 5, Corollary 1] introduce the extended Inada condition

$$\lim_{k \downarrow 0} [-k f''(k)] > 1. \tag{GR}$$

The shocks (ε_t) are uncorrelated and identically distributed on $\mathbb{R}_+$ according to density ψ. We assume that ψ is strictly positive on $\mathbb{R}_+$.

As Galor and Ryder point out [22, Lemma 1, p. 365], restrictions on the utility function are necessary to obtain unique self-fulfilling expectations. Here it is assumed that agents maximize utility

$$U(c_t, c'_{t+1}) = \ln c_t + \beta \mathbb{E}(\ln c'_{t+1}), \quad \beta \in (0, 1),$$

subject to the budget constraint $c'_{t+1} = (w_t - c_t)(1 + r_{t+1})$, where c (respectively, c') is consumption while young (respectively, old), w_t is the wage and r_t is the interest rate. In this case optimization implies a savings rate from wage income of $\beta/(1+\beta)$, whence $k_{t+1} = (\beta/(1+\beta))w_t$. Assuming that labor is paid its marginal factor product yields the law of motion

$$k_{t+1} = \frac{\beta}{1+\beta}[f(k_t) - k_t f'(k_t)]\varepsilon_t. \tag{4.2}$$

Proposition 4.1. *Assume that the Galor-Ryder condition* (GR) *holds. If, in addition,* $\mathbb{E}(\varepsilon) < \infty$ *and* $\mathbb{E}(1/\varepsilon) < \beta/(1+\beta)$*, then the economy* (4.2) *has a unique and globally stable stochastic equilibrium.*

Remark 4.1. As in Chapter 3, Section 3.3.2, the bound $\mathbb{E}(1/\varepsilon) < \beta/(1+\beta)$ is used to restrict weight in the left-hand tail of ψ, preventing the economy from collapsing to zero as a result of adverse shocks.

Proof. We verify that (4.2) satisfies the conditions of Corollary 4.1. To this end, let $D = \beta/(1+\beta)$, let $h(k) = f(k) - kf'(k)$ and let $g(k) = Dh(k)$. It follows from our assumptions on f that the function $k \mapsto h(k)$ is zero at zero, continuously differentiable, strictly increasing and satisfies $\lim_{k\downarrow 0} h'(k) > 1$. This last fact—which is equivalent to (GR)—implies that

$$\exists\, \delta > 0 \text{ s.t. } h(k) \geq k, \quad \forall k \in [0, \delta). \tag{4.3}$$

Evidently Assumptions 4.1 and 4.2 are satisfied. Regarding Condition 4.1, consider the Lyapunov function defined by $V(k) = 1/k + k$. We have

$$\int V[g(k)z]\psi(z)dz = \mathbb{E}(1/\varepsilon)\frac{1}{g(k)} + \mathbb{E}(\varepsilon)g(k). \tag{4.4}$$

Consider the first term in the right hand side of (4.4). By (4.3),

$$\mathbb{E}(1/\varepsilon)\frac{1}{g(k)} \leq \alpha_1 \frac{1}{k}, \quad \forall k \in [0, \delta), \tag{4.5}$$

where $\alpha_1 = \mathbb{E}(1/\varepsilon)D^{-1} < 1$. In addition, monotonicity of g yields

$$\mathbb{E}(1/\varepsilon)\frac{1}{g(k)} \leq \mathbb{E}(1/\varepsilon)\frac{1}{g(\delta)}, \quad \forall k \in [\delta, \infty). \tag{4.6}$$

Combining (4.5) and (4.6) gives

$$\mathbb{E}(1/\varepsilon)\frac{1}{g(k)} \leq \alpha_1 \frac{1}{k} + C_1, \quad \forall k \in \mathbb{R}_+. \tag{4.7}$$

Consider now the second term in the sum (4.4). By the assumptions on f it is clear that the function $k \mapsto \mathbb{E}(\varepsilon)Df(k)$ can be majorized on $\mathbb{R}_+$ by an affine function $k \mapsto \alpha_2 k + C_2$, where α_2 and C_2 are nonnegative constants, $\alpha_2 < 1$. Therefore,

$$\mathbb{E}(\varepsilon)g(k) \leq \mathbb{E}(\varepsilon)Df(k) \leq \alpha_2 k + C_2, \quad \forall k \in \mathbb{R}_+. \tag{4.8}$$

Let $\alpha = \max(\alpha_1, \alpha_2)$, and let $C = C_1 + C_2$. Substituting (4.7) and (4.8) into (4.4) gives

$$\int V[g(k)z]\psi(z)dz \leq \alpha(1/k + k) + C = \alpha V(k) + C. \tag{4.9}$$

Since $\alpha < 1$, Condition 4.1 is satisfied.

In addition, Condition 4.2 is satisfied by hypothesis, and $k \mapsto g(k)$ is monotone increasing on $\mathbb{R}_+$. Thus all of the conditions of Corollary 4.1 are verified. □

4.3.2 Stability in a Model with Externalities

Consider the following Brock-Mirman optimal growth model with increasing returns. The production function is Cobb-Douglas with an external effect due to the existence of increasing social returns; technology is sensitive to economy-wide capital aggregates. Thus,

$$y_{t+1} = A(k_t) k_t^{\alpha} \ell_t^{1-\alpha} \varepsilon_t, \tag{4.10}$$

where externalities are captured by the function $k \mapsto A(k)$. This dependence is external to individual agents, and A is treated as constant with respect to private investment. The capital share α satisfies $0 < \alpha < 1$.

Regarding the nature of increasing returns, we assume only that

Assumption 4.3. The range of $A \colon \mathbb{R}_+ \to \mathbb{R}_+$ is contained in a closed and bounded subset of $\mathbb{R}_+ \setminus \{0\}$.

Macroeconomic models with external effects satisfying Assumption 4.3 include Azariadis and Drazen [6], Galor and Zeira [23], and Quah [48]. For example, Azariadis and Drazen consider the model

$$A(k) = \begin{cases} A_1, & \text{if } k \leq k_b; \\ A_2, & \text{if } k > k_b. \end{cases}$$

Here $0 < A_1 < A_2$ represent the state of technology, and k_b is a fixed "threshold" level of capital per worker.

For convenience, labor supply is normalized to unity. The productivity shocks $\varepsilon_t \in \mathbb{R}_+$ are uncorrelated and identically distributed with density ψ.

Let c_t be time t consumption. A representative agent seeks to maximize

$$\mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t \ln c_t\right], \quad \beta \in (0,1),$$

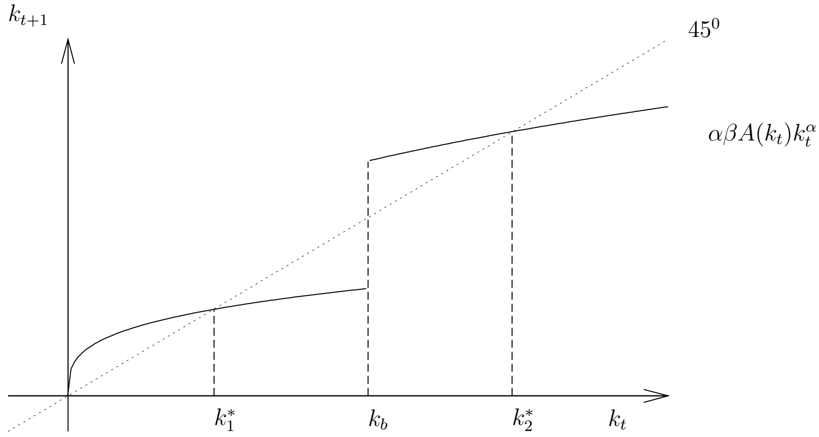

Figure 4.1: The map $k_t \mapsto \alpha\beta A(k_t)k_t^\alpha$.

subject to $k_{t+1} + c_t \leq y_t$. The optimal policy [58, p. 19 and pp. 274–77] is

$$k_{t+1} = \alpha\beta A(k_t)k_t^\alpha \varepsilon_t. \tag{4.11}$$

A plot of the map $k \mapsto \alpha\beta A(k)k^\alpha$ is given in Figure 4.1 for the Azariadis-Drazen threshold case, in which A is a step function. As drawn, the deterministic version has two equilibria, k_1^* and k_2^*.

Despite the apparent simplicity of (4.11), establishing dynamic stability is complicated by the dependence of A on k_t, which is potentially highly nonlinear. For example, *none* of the three main sufficient conditions used by Stokey, Lucas and Prescott [58, Theorem 12.12] are satisfied. Also, in the deterministic case (when ε_t is held constant), the model (4.11) may have multiple local attractors. It is therefore somewhat surprising that

Proposition 4.2. *For a class of shocks that include the lognormal distributions, the economy* (4.11) *has a unique, globally stable stochastic equilibrium.*

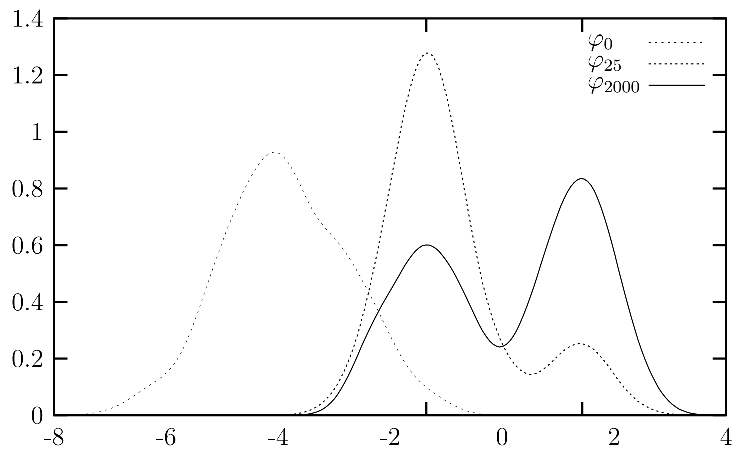

Figure 4.2: Convergence to equilibrium

Proof. We verify that (4.11) satisfies the conditions of Theorem 4.1. Evidently Assumptions 4.1 and 4.2 hold. By the hypotheses of the proposition, we may assume that ψ satisfies Conditions 4.2 and 4.3, and that $\mathbb{E}\,|\ln \varepsilon| < \infty$. Regarding Condition 4.1, let $V(k) = |\ln k|$. The function V so constructed is a Lyapunov function on $\mathbb{R}_+$. Moreover,

$$\begin{aligned}\int V[DA(k)k^\alpha z]\psi(z)dz &= \int |\ln D + \ln A(k) + \alpha \ln k + \ln z|\psi(z)dz \\ &\leq \alpha|\ln k| + C \\ &= \alpha V(k) + C,\end{aligned}$$

where $C = |\ln D| + \sup_k |\ln A(k)| + \mathbb{E}\,|\ln \varepsilon|$. Since $\alpha < 1$ and $C < \infty$, Condition 4.1 holds. The proof is complete. $\square$

To illustrate this result, Figure 4.2 presents a sequence of densities generated by iterating the Markov operator P implied by (4.11)—see Section 4.4 for details on the construction of P—on an arbitrary initial distribution φ_0, where the x-axis is the logarithm of capital per head.[4] Here φ_0 can be thought of as an initial distribution of a "large" number of Azariadis-Drazen economies.

In the figure, the density φ_0 is the left-most distribution, with probability mass shifting rightwards over time. It is interesting to observe that the nonlinear external effects in the production function (4.10) lead to the cross-country income distribution developing a bimodal structure which has been observed in the actual cross-country growth data by, among others, Quah [48, 49], Jones [31] and Durlauf and Quah [16].

Proposition 4.2 implies that the sequence of densities (φ_n) converges to a unique limiting density φ^*. In this case there is little observable change after $t = 2000$.

4.3.3 Existing Conditions

Previously a set of conditions for obtaining stability of the model (4.1) was identified by K. Horbacz [30, Theorem 1]. Her results can be obtained as a special case of Theorem 4.2.

The statement and proof of the above problem are somewhat tangential to the economics (as opposed to mathematics) discussed in this chapter. As a

[4] The parameters are $\beta = 1$, $\alpha = 0.5$, $A_1 = 0.5$, $A_2 = 2$, $k_b = 0.6875$, ε lognormal, $\ln \varepsilon \sim N(0, 0.5)$. The densities are generated using a Monte Carlo simulation and estimated nonparametrically by the Parzen window method with Gaussian kernel and bandwidth 0.38. Such estimates are known to converge to the true density in L_1 norm for large sample size [12]. Here each generation is represented by 200 sample points.

result they have been deferred to the appendix.

4.4 Formulation of the Problem

Prior to the proofs, a more formal definition of the model (4.1) is given. To begin, let $\mathbb{R}$ be the real numbers, let $\mathcal{B}$ be the Borel sets of $\mathbb{R}$, let $\mathbb{R}_+ = [0, \infty)$, and let $\mathcal{B}_+ = \mathcal{B} \cap \mathbb{R}_+$. The Lebesgue measure is again denoted by μ. Integration where the measure is not made explicit is taken with respect to μ; integration using the symbol $\int$ without subscript is taken over $\mathbb{R}_+$.

In what follows we use the notation of Chapter 2, Section 2.3, where the underlying space X of Chapter 2 now corresponds to the half-ray $\mathbb{R}_+$ with the usual topology.

Let the sequence of shocks (ε_t) in (4.1) be uncorrelated and identically distributed by $\Psi \in \mathcal{P}$. We assume that Ψ is absolutely continuous with respect to μ. In this case there exists a unique density $\psi \in D(\mu)$ satisfying $\int_B \psi = \Psi(B)$ for all $B \in \mathcal{B}_+$; ψ is the Radon-Nikodým (RN) derivative of Ψ with respect to μ.

Definition 4.1. Let $g \colon \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. In what follows, a *perturbed dynamical system* on $\mathbb{R}_+$ is defined by a pair (g, ψ), where, given current state value $x_t \in \mathbb{R}_+$, a shock $\varepsilon_t \in \mathbb{R}_+$ is selected independently from density ψ, and the next period state is realized as in (4.1).

We wish to embed the Markov chain generated by (g, ψ) in the function space $L_1(\mu)$. To do so requires that the transition probabilities $B \mapsto \mathbf{N}(x, B)$ of the Markov chain (see (2.8), p. 26) can be represented by density functions (recall Assumption 2.1, p. 30, and the subsequent discussion).

If g satisfies Assumption 4.2 and ε is distributed according to density ψ, then for almost all x, the distribution $B \mapsto \mathbf{N}(x, B)$ is absolutely continuous with respect to μ (i.e., Assumption 2.1 is satisfied). To see this, pick any x such that $g(x) > 0$, and any $E \in \mathcal{B}$ such that $\mu(E) = 0$. We need to show that $\mathbf{N}(x, E) = 0$. This must be the case, because if E is a null set then so is $E/g(x)$, and hence

$$\begin{aligned} \mathbf{N}(x, E) &= \int \mathbf{1}_E[g(x)z]\psi(z)dz \\ &= \int_{E/g(x)} \psi(z)dz \\ &= 0. \end{aligned}$$

In particular, for x such that $g(x) > 0$,

$$p(x, y) = \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}, \tag{4.12}$$

because changing variables shows that for any $B \in \mathcal{B}_+$,

$$\int_B p(x, y)dy = \int \mathbf{1}_B[g(x)z]\psi(z)dz = \mathbf{N}(x, B).$$

For other x set $p(x, \cdot)$ equal to any density.[5]

It follows from this discussion that the density kernel p exists, as does the Markov operator in $L_1(\mu)$ defined by (2.11) on page 30.

[5]Density kernels need be defined only up to the complement of a null set—systems with kernels equal $\mu \times \mu$-a.e. have identical Markov operators and we do not distinguish between them in what follows.

The notion of existence, uniqueness and stability of equilibrium used in the essay can now be formalized by Definition 2.4, page 31.

4.5 Proofs

The main proof is based on Theorem 2.2, page 38. We show that the semidynamical system $(D(\mu), P)$ associated with the economy (g, ψ) is both strongly contracting (Definition 2.6, p. 36) and Lagrange stable (Definition 2.5, p. 33). As usual, $D(\mu)$ is treated as a metric space with the L_1 norm distance.

Lemma 4.1. *Let (g, ψ) be a perturbed dynamical system satisfying Assumptions 4.1 and 4.2, let p be the density kernel defined in Section 4.4, and let P be the Markov operator associated with p by* (2.11)*. If Condition 4.2 holds, then the semidynamical system $(D(\mu), P)$ is strongly contracting.*

Proof. Since ψ is strictly positive, representation (4.12) implies that the density kernel p is $\mu \times \mu$-a.e. strictly positive on $X \times X$. As stated earlier, perturbed dynamical systems with kernels that are equal $\mu \times \mu$-a.e. have identical Markov operators. If p is strictly positive, then $(D(\mu), P)$ is strongly contracting by Lemma 2.3, page 39. The result follows. $\square$

Next we treat Lagrange stability of the semidynamical system $(D(\mu), P)$ associated with (g, ψ). Recall that to prove Lagrange stability it is sufficient to find a set $\mathcal{D} \subset D(\mu)$ such that $\mathcal{D}$ is norm-dense in $D(\mu)$ and the trajectory of φ under P is weakly precompact for each $\varphi \in \mathcal{D}$ (Proposition 2.1, p. 41).

Lemma 4.2. *Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 4.1 and 4.2, and let P be the associated Markov operator. If Condition 4.1 and either one of Condition 4.3 or 4.4 holds, then there exists a set $\mathcal{D} \subset L_1(\mu)$ such that $\mathcal{D}$ is dense in $D(\mu)$ and $\{P^t\varphi : t \geq 0\}$ is weakly precompact for each $\varphi \in \mathcal{D}$.*

Proof. Let V be the Lyapunov function in Condition 4.1. Let $\mathcal{D}$ be the set of all density functions φ in $L_1(\mu)$ such that

$$\int V(x)\varphi(x)dx < \infty. \tag{4.13}$$

We claim that $\mathcal{D}$ has the desired properties.

Pick any density φ. To see that there exists a $(\varphi_k) \subset \mathcal{D}$ with $\varphi_k \to \varphi$, define first $\varphi_k^0 = \mathbf{1}_{[1/k,k]}\varphi$. By the monotone convergence theorem, $\|\varphi_k^0\| \to 1$. Hence $\|\varphi_k^0\| > 0$ for all k greater than some constant K. For all such k define $\varphi_k = \|\varphi_k^0\|^{-1}\varphi_k^0$. Then $\varphi_k \in D(\mu)$ for all $k \geq K$ by construction. Moreover, $\varphi_k \to \varphi$ pointwise, and hence in L_1 norm by Scheffé's lemma. Finally, $\varphi_k \in \mathcal{D}$ for all $k \geq K$, because

$$\int V(x)\varphi_k(x)dx = \frac{1}{\|\varphi_k^0\|}\int \mathbf{1}_{[1/k,k]}(x)V(x)\varphi(x)dx,$$

and V is bounded on compact subsets of $\mathbb{R}_+ \setminus \{0\}$ by continuity.

It remains to show that if $\varphi \in \mathcal{D}$, then $\{P^t\varphi : t \geq 0\}$ is weakly precompact. Note first that the collection $\{P^t\varphi\}$ is norm-bounded, because $PD(\mu) \subset D(\mu)$. Thus it remains only to verify parts (i) and (ii) of the Dunford-Pettis condition (Section 2.5.3, p. 41).

Regarding (i), pick any $\varepsilon > 0$. We exhibit a $\delta > 0$ and an $N \in \mathbb{N}$ such that

$$\mu(A) < \delta \implies \int_A P^t f(x)dx < \varepsilon, \quad \forall\, t \geq N.$$

Define $E(V|g) = \int Vg$. By Fubini's theorem,

$$\begin{aligned} E(V|P^t\varphi) &= \int V(y)P^t\varphi(y)dy \\ &= \int V(y)\left[\int p(x,y)P^{t-1}\varphi(x)dx\right]dy \\ &= \int \left[\int V(y)p(x,y)dy\right]P^{t-1}\varphi(x)dx. \end{aligned}$$

But

$$\int V(y)p(x,y)dy = \int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C$$

for all x by Condition 4.1. Therefore,

$$E(V|P^t\varphi) \leq \int [\alpha V(x) + C]P^{t-1}\varphi(x)dx = \alpha E(V|P^{t-1}\varphi) + C.$$

Repeating this argument obtains

$$E(V|P^t\varphi) \leq \alpha^n E(V|\varphi) + \frac{C}{1-\alpha}.$$

Since $E(V|\varphi)$ is finite by (4.13), it follows that

$$E(V|P^t\varphi) \leq 1 + \frac{C}{1-\alpha}, \quad t \geq N,$$

for some $N \in \mathbb{N}$.

On the other hand, it can be verified that for arbitrary positive a,

$$a\int_{\mathbb{R}_+ \setminus G_a} P^t\varphi \leq E(V|P^t\varphi)$$

when G_a is defined as the set of $x \in \mathbb{R}_+$ with $V(x) \leq a$. Therefore,

$$\int_{\mathbb{R}_+ \setminus G_a} P^t \varphi \leq \frac{1}{a}\left(1 + \frac{C}{1-\alpha}\right), \quad \forall\, t \geq N, \quad \forall\, a > 0. \tag{4.14}$$

Choose a so large that

$$\frac{1}{a}\left(1 + \frac{C}{1-\alpha}\right) \leq \frac{\varepsilon}{2}. \tag{4.15}$$

Consider now the decomposition

$$\int_A P^t \varphi = \int_{A \cap G_a} P^t \varphi + \int_{A \cap [\mathbb{R}_+ \setminus G_a]} P^t \varphi.$$

Using (4.14) and (4.15) gives

$$\int_A P^t \varphi \leq \int_{A \cap G_a} P^t \varphi + \frac{\varepsilon}{2}, \tag{4.16}$$

whenever $t \geq N$. Here a is the constant determined in (4.15).

The next step is to bound the first term in the sum on the right hand side of (4.16), taking the constant a as given, and assuming that at least one of Condition 4.3 or Condition 4.4 holds.

Assume first that Condition 4.3 holds. Using the expression for the density kernel given in (4.12), we have

$$\begin{aligned}
P^t \varphi(y) &= \int p(x,y) P^{t-1} \varphi(x) dx \\
&= \int \psi\left(\frac{y}{g(x)}\right) \frac{1}{g(x)} P^{t-1} \varphi(x) dx \\
&= \int \psi\left(\frac{y}{g(x)}\right) \frac{y}{g(x)} \frac{1}{y} P^{t-1} \varphi(x) dx \\
&\leq \frac{M}{y}.
\end{aligned}$$

Therefore,

$$\int_{A \cap G_a} P^t \varphi(y) dy \leq \int_{A \cap G_a} \frac{M}{y} dy \leq \int_A J(a) dy = J(a)\mu(A),$$

where the finite number $J(a)$ is the maximum of M/y over the closed and bounded interval $G_a \subset \mathbb{R}_+ \setminus \{0\}$.

Now pick any positive δ satisfying $\delta \leq \varepsilon/(J(a)2)$. For such a δ we have

$$\mu(A) < \delta \implies \int_{A \cap G_a} P^t f(x)dx < \frac{\varepsilon}{2}.$$

Combining this with (4.16) proves (i) of the Dunford-Pettis characterization for the collection $\{P^t\varphi : t \geq N\}$ when Condition 4.3 holds.

We now establish the same when Condition 4.4 holds, again by bounding the first term in the sum (4.16). Suppose first that there exists a c with $g(x) \geq c > 0$ for *all* x in $\mathbb{R}_+$. In this case, because

$$\begin{aligned}\int_{A \cap G_a} P^t\varphi(y)dy &= \int_{A \cap G_a} \int p(x,y)P^{t-1}\varphi(x)dxdy \\ &= \int \left[\int_{A \cap G_a} p(x,y)dy \right] P^{t-1}\varphi(x)dx,\end{aligned}$$

and because

$$\begin{aligned}\int_{A \cap G_a} p(x,y)dy &= \int_{A \cap G_a} \psi\left(\frac{y}{g(x)}\right)\frac{1}{g(x)}dy \\ &= \int_{\frac{A \cap G_a}{g(x)}} \psi(z)dz \\ &\leq \int_{\frac{A \cap G_a}{c}} \psi(z)dz \\ &\leq \int_{\frac{A}{c}} \psi(z)dz\end{aligned}$$

for all $x \in \mathbb{R}_+$, it follows that if $\delta' > 0$ is chosen such that

$$\mu(A) < \delta' \implies \int_A \psi(z)dz < \frac{\varepsilon}{2}$$

(existence of such a δ' is by absolute continuity of $A \mapsto \int_A \psi$ with respect to μ), then

$$\int_{A \cap G_a} p(x,y)dy \leq \int_{\frac{A}{c}} \psi(z)dz < \frac{\varepsilon}{2}$$

whenever $\mu(A) < \delta$, $\delta = \delta' c$, and, therefore,

$$\mu(A) < \delta \implies \int_{A \cap G_a} P^t \varphi < \frac{\varepsilon}{2}.$$

Again, combining this with (4.16) yields (i) of the Dunford-Pettis condition.

Finally, suppose to the contrary that while Condition 4.4 is satisfied, there exists no c with $g(x) \geq c > 0$ for all $x \in \mathbb{R}_+$. In this case Condition 4.4 implies that $g(x) \downarrow 0$ as $x \downarrow 0$, and hence there exists a $d > 0$ such that

$$\int_{A \cap G_a} p(x,y)dy = \int_{\frac{A \cap G_a}{g(x)}} \psi(z)dz \leq \frac{\varepsilon}{2} \quad \text{for almost all } x \in [0,d), \tag{4.17}$$

owing to the fact that $A \cap G_a$ is bounded away from 0. For $x \geq d$, $g(x) \geq c' = \min[g(d), b] > 0$, where b is the positive constant in Condition 4.4.[6] In this case, an argument similar to that given above for the case $g(x) \geq c > 0$ implies that

$$\int_{A \cap G_a} p(x,y)dy \leq \int_{\frac{A}{c'}} \psi(z)dz < \frac{\varepsilon}{2} \tag{4.18}$$

whenever $x \in [d, \infty)$ and $\mu(A) < \delta$, $\delta = \delta' c'$. Combining (4.17) and (4.18) yields

$$\mu(A) < \delta \implies \int_{A \cap G_a} P^t \varphi < \frac{\varepsilon}{2}.$$

Once again, (i) of the Dunford-Pettis characterization holds.

It remains to establish that part (ii) of the Dunford-Pettis condition also holds for the same collection. We have already shown that

$$\int_{\mathbb{R}_+ \setminus G_a} P^t \varphi \leq \frac{1}{a} \left(1 + \frac{C}{1 - \alpha} \right)$$

for all positive a, all $t \geq N$. But this inequality is sufficient, because G_a is always bounded. Hence condition (ii) is also satisfied for $\{P^t \varphi \,:\, t \geq N\}$. This completes the proof of the lemma. □

[6]Here $g(d) > 0$ by Condition 4.4 and the almost everywhere positivity of g.

Chapter 5

Asymptotic Distributions

5.1 Introduction

As discussed in Chapter 3, dynamic properties of the one-sector stochastic optimal growth model with concave production technology were first studied in the well-known paper of Brock and Mirman [10]. Using the same assumptions on preferences and technology as used in the deterministic case, they showed that there exists a unique and globally stable stochastic equilibrium. In their framework of analysis the influence of the shock is bounded, and the state eventually converges to a compact "invariant set" in the positive real numbers.

Stochastic models that have an equilibrium or steady state distribution may also have asymptotic statistical properties related to the existence of a steady state distribution, such as convergence of sample averages from time series to the mean of this limiting distribution, or asymptotic normality of the partial sums. The first question concerns a law of large numbers (LLN) result, while the latter concerns a central limit theorem (CLT) result.

The importance of these questions can be summarized as follows. If an LLN condition holds, then it is possible to test a given theoretical model by comparing the mean of the limiting distribution with a sample average from a sufficiently large data set generated by the system under study. Conversely, suppose that an expression is available for the mean of the hypothetical model in terms of its parameters. Then the implied equality of this expression and the sample mean calculated from data provides a consistent method for estimating parameter values. If, in addition, a CLT result is available, then inference can be drawn as to the likelihood of values in the parameter space.

It known that both the LLN and the CLT result are realized for the general discrete-time concave stochastic optimal growth model when the shock has compact support [58, 17, 5, 7]. Central to the proofs is boundedness of the productivity shock, which, in combination with Inada conditions, allow the state space to be taken to be compact.[1] When the influence of the shock is not bounded, however, compactness of the state space fails. In this case it is unclear whether or not LLN and CLT properties continue to hold.

In this essay we begin to address this question by introducing a methodology and studying some specific parameterizations. The techniques are based on results recently obtained in an important paper of Łoskot and Rudnicki [47],

[1] In a related paper, ergodicity in moments for the Solow-Swan model with a shock that is unbounded above but cannot be arbitrarily small is investigated in Binder and Pesaran [8].

which studies the LLN and CLT properties of nonlinear dynamical systems perturbed by uncorrelated noise.[2] The question of asymptotic statistical properties for the general concave one-sector optimal growth model with unbounded shock is left open. It is hoped that the methodology used here can be extended to the general problem.

Section 5.2 formulates the problem and gives the major definitions. Section 5.3 provides a general stability result. Section 5.4 gives applications. Section 5.5 gives proofs.

5.2 The Model

As in Chapter 2, we consider a growth model evolving on state space X, where X is an arbitrary topological space. Specifically, we consider the stochastic dynamic economy (T, Ψ) of Chapter 2, Section 2.3.1.

As in Chapter 2, Section 2.3, $\mathcal{B}$ is the Borel sets of X, $\mathcal{M}$ denotes the normed vector lattice of finite signed Borel measures $\mu\colon \mathcal{B} \to \mathbb{R}$, and $\mathcal{P}$ is the distributions in $\mathcal{M}$. All integrals of real functions defined on X are over the whole space X unless otherwise stated.

In this chapter, however, we assume throughout that the topology of X is metrizable. Let $\varrho\colon X \times X \to \mathbb{R}_+$ be a distance which metrizes the topology on X.

[2] Previously, the techniques of Łoskot and Rudnicki have been applied to areas such as neurodynamics [19] and entropy computation in iterated function systems [53].

To briefly recall the main ideas from Chapter 2, random outcomes are selected from some measurable space $(\Omega, \mathcal{F})$ by probability measure $\mathbf{P}$, and mapped into X by random variable $\varepsilon \colon \Omega \to X$. Corresponding to ε is a finite Borel measure $\Psi \in \mathcal{P}$ defined at $B \in \mathcal{B}$ by $\Psi(B) = \mathbf{P}[\varepsilon^{-1}(B)]$. The measure Ψ is called the *distribution* of ε, and, as usual, the distribution satisfies $\int_\Omega f[\varepsilon(\omega)]\mathbf{P}(d\omega) = \int_X f(z)\Psi(dz)$ for any real $\mathcal{B}$-measurable function f.

Given a transition rule T mapping $X \times X$ into X, and, given current state value $x_t \in X$, a shock $\varepsilon_t \in X$ is selected independently from Ψ, and the next period state is realized as

$$x_{t+1} = T(x_t, \varepsilon_t). \tag{5.1}$$

The notion of equilibrium we use is again the standard one of Chapter 2 (Definition 2.3, p. 28). To repeat, an *equilibrium* for the economy (T, Ψ) is a probability measure $\varphi \in \mathcal{P}$ that satisfies

$$\int \left[\int \mathbf{1}_B[T(x,z)]\Psi(dz) \right] \varphi(dx) = \varphi(B) \tag{5.2}$$

for all $B \in \mathcal{B}$.[3] The equilibrium is unique if there exists no other point in $\mathcal{P}$ satisfying (5.2).

Suppose that the growth model (5.1) has a unique equilibrium φ. For the purposes of this paper, (5.1) is said to satisfy the law of large numbers if, for any Lipschitz function $g \colon X \to \mathbb{R}$,

$$\frac{1}{N} \sum_{t=0}^{N-1} g(x_t) \to \int g(x)\varphi(dx) \tag{5.3}$$

[3]As before, $\mathbf{1}_B \colon X \to \{0, 1\}$ is the characteristic function of $B \in \mathcal{B}$.

P-almost surely as $N \to \infty$.[4]

The economy is said to have the central limit property if, for any g as above,

$$\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} g(x_t) \to N(m, \sigma^2) \tag{5.4}$$

in distribution, where $N(m, \sigma^2)$ is a normal distribution with mean $m = \int g(x)\varphi(dx)$ and variance $\sigma^2 \geq 0$.

5.3 Results

Łoskot and Rudnicki [47] consider stochastic models that satisfy the following contraction condition.

Definition 5.1. The pair (T, Ψ), where T is the map in (5.1) and Ψ is the distribution of the shock ε, is called an *average contraction* with respect to ϱ if there exists a Borel function $\lambda \colon X \to \mathbb{R}$ such that $E(\lambda) = \int \lambda(z)\Psi(dz) < 1$ and

$$\varrho(T(x, z), T(x', z)) \leq \lambda(z)\varrho(x, x'), \quad \forall x, x', z \in X.$$

By adapting the results of Łoskot and Rudnicki and using additional restrictions on the space X, we obtain the following stability condition.

Theorem 5.1. *Let (T, Ψ) be the stochastic dynamic economy of Section 5.2. Let the state space X be both locally compact and σ-compact in the topology*

[4] A real function g on X is called *Lipschitz* if there exists a constant λ such that $|g(x) - g(x')| \leq \lambda\varrho(x, x')$ for any $x, x' \in X$.

metrized by ϱ.[5] *If the growth model defined by the law of motion* T *and the distribution of the shock* Ψ *is an average contraction with respect to* ϱ, *and if there exists at least one point* $\bar{x} \in X$ *such that*

$$\int \varrho(\bar{x}, T(\bar{x}, z))\Psi(dz) < \infty, \tag{5.5}$$

then there exists a unique stochastic equilibrium $\varphi \in \mathcal{P}$ *satisfying* (5.2), *and, in addition, the model satisfies the law of large numbers property* (5.3). *If, moreover,* $\mathbb{E}(\lambda^2) < 1$ *and*

$$\int [\varrho(\bar{x}, T(\bar{x}, z))]^2 \Psi(dz) < \infty \tag{5.6}$$

holds, then the model also satisfies the central limit property (5.4).

The proof (Section 5.5) is a straightforward consequence of the results of Łoskot and Rudnicki. The only technical difficulty is to verify that the steady state notion used by Łoskot and Rudnicki is equivalent to the definition (5.2), which is standard in the economic literature. This can be done under local and σ-compactness of the state space, as was assumed in the theorem.

5.4 Applications

Let $X = (0, \infty)$. Consider the one-sector optimal growth problem

$$\max \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{5.7}$$

$$\text{s.t. } k_{t+1} = f(k_t, \varepsilon_t) - c_t \tag{5.8}$$

[5] Recall that a topological space is called *locally compact* if every point in the space has a neighborhood with compact closure, and *σ-compact* if every open set can be obtained as the union of a countable number of compact sets.

where $c_t \in X$ is consumption, $k_t \in X$ is capital per head, $\beta \in (0,1)$ is a discount factor, and $f\colon X \times X \to X$ and $u\colon X \to \mathbb{R}$ are the production and utility functions respectively [10, pp. 484–488]. The utility function u is assumed to satisfy $u'(0) = \infty$, which assures interiority of solutions, and therefore eliminates the possibility of zero savings or consumption. The shocks $\varepsilon_t \in X$ are independent draws from $\mathbf{P}$ as before.

The solution to the planning problem, if it exists, is an optimal policy $g\colon X \to X$, which associates realized output $f(k_t, \varepsilon_t)$ with optimal current consumption c_t. Optimal consumption $g(f(k_t, \varepsilon_t))$ can then be substituted into (5.8) to obtain the closed loop law of motion for the system, which is in the form of (5.1). A unique and well-defined stochastic process is specified by this law and any initial condition $k_0 \in X$. The process so generated is called an *optimal program*.

Example 5.1. Consider first the unit-elastic decreasing returns model $u(c) = \ln c$, $f\colon (k, \varepsilon) \mapsto Ak^\alpha \varepsilon$, $A > 0$, $\alpha \in (0,1)$. For such a specification, the optimal policy consumes a fraction $1 - \alpha\beta$ of realized output $Ak^\alpha\varepsilon$, implying the law of motion

$$k_{t+1} = \alpha\beta A k_t^\alpha \varepsilon_t. \tag{5.9}$$

Define a binary ϱ on $X \times X$ by $\varrho(x, y) = |\ln x - \ln y|$. Evidently ϱ is a metric on X. Moreover, the space (X, ϱ) is isometrically isomorphic to $\mathbb{R}$ under the mapping $x \mapsto \ln x$ when the latter space is endowed with its usual Euclidean metric. Hence (X, ϱ) is both locally and σ-compact.

It can be verified that (5.9) is an average contraction on (X, ϱ) for every random variable ε. If $\mathbb{E}|\ln \varepsilon|$ is finite, then condition (5.5) of Theorem 5.1

holds for $\bar{x} = 1$, implying the existence of a unique stochastic equilibrium $\varphi \in \mathcal{P}$, and the LLN result (5.3) for the process $(k_t)_{t\geq 0}$.

Evidently $\mathbb{E}(\lambda^2) < 1$ also holds. If, in addition, $\mathbb{E}[(\ln \varepsilon)^2]$ is finite, then (5.6) is satisfied for $\bar{x} = 1$, and the CLT result (5.4) obtains.

Remark 5.1. The conditions $\mathbb{E}|\ln \varepsilon| < \infty$ and $\mathbb{E}[(\ln \varepsilon)^2] < \infty$ can be viewed as restrictions on the left- and right-hand tails of the distribution. See the discussion in Stachurski [55, Remark 4.1].

Example 5.2. The second example is from Mirman and Zilcha [46, Example A, p. 333]. The state space X, the shock ε, the discount factor β, the productivity parameter A and utility $u(c) = \ln c$ are as before. Let α be a Borel function from X into $(0, 1)$. The production function is $(k, \varepsilon) \mapsto Ak^{\alpha(\varepsilon)}$. For such a specification, the law of motion is

$$k_{t+1} = \bar{\alpha}\beta A k_t^{\alpha(\varepsilon_t)}, \quad \bar{\alpha} = \int_\Omega \alpha[\varepsilon(\omega)]\mathbf{P}(d\omega). \tag{5.10}$$

Once again, the system is an average contraction on X under the metric ϱ, this time using $\lambda(z) = \alpha(z)$. Note that in this case (5.5) holds for *any* shock ε when $\bar{x} = 1$. Hence for any strictly positive initial condition k_0, a unique equilibrium distribution φ exists and the LLN condition (5.3) holds.

Moreover, $\mathbb{E}(\lambda^2) < 1$, and (5.6) holds for any shock ε when $\bar{x} = 1$, implying that the CLT condition (5.4) also holds.

5.5 Proofs

This section contains the proof of Theorem 5.1. Throughout, C_b denotes the Banach lattice of continuous bounded real functions on X, and $C_0 \subset C_b$ denotes the continuous real functions with compact support. In what follows, the scalar product notation is used for integration. Thus, for bounded Borel function $f\colon X \to \mathbb{R}$ and $\mu \in \mathcal{M}$, we write $\langle f, \mu \rangle$ for $\int f d\mu$. Finally, let $\mathcal{M}_+$ be the nonnegative measures in $\mathcal{M}$ (i.e., $\mathcal{M}_+$ is the positive cone of $\mathcal{M}$).

For $f \in C_0$ and $\mu \in \mathcal{M}_+$,

$$\Lambda(f) = \langle f, \mu \rangle \tag{5.11}$$

defines a positive linear functional Λ on C_0. Denote by C_0^* the class of positive linear functionals on C_0. We will make use of the well-known fact that the association $\mu \mapsto \Lambda$ from the finite Borel measures $\mathcal{M}_+$ to the linear functionals C_0^* defined by (5.11) is one-to-one [2, Theorems 38.3 and 38.4].

Łoskot and Rudnicki [47, Theorems 1 and 3] proved that when (T, Ψ) is an average contraction, (X, ϱ) is complete and separable, and (5.5)–(5.6) holds, then there exists a unique distribution $\varphi \in \mathcal{P}$ such that

$$\int \left[\int f[T(x,z)] \Psi(dz) \right] \varphi(dx) = \int f(x) \varphi(dx), \quad \forall f \in C_b, \tag{5.12}$$

and, moreover, the LLN and CLT results (5.3) and (5.4) both hold for φ.

Every locally and σ-compact metric space is both complete and separable. Thus to establish Theorem 5.1 it suffices to verify that the condition (5.12) characterizes the set of Brock-Mirman equilibria under the hypotheses of the theorem.

Lemma 5.1. *Let* (X, ϱ) *be a locally and* σ*-compact metric space, and let* φ *be any finite Borel measure on* X*. The measure* φ *satisfies (5.12) if and only if it also satisfies (5.2).*

Proof. In what follows, by a *Borel function* is meant a real $\mathcal{B}$-measurable function on X. Define an operator P^* from the set of all bounded Borel functions $f\colon X \to \mathbb{R}$ into itself by

$$(P^*f)(x) = \int f[T(x,z)]\Psi(dz), \quad x \in X.$$

Define in addition an operator P from the space of finite measures $\mathcal{M}_+$ into itself by

$$(P\mu)(B) = \int\int \mathbf{1}_B[T(x,z)]\Psi(dz)\mu(dx), \quad B \in \mathcal{B}.$$

(This is just the Markov operator associated with (T, Ψ).) The operator P is "adjoint" to P^*, in the sense that

$$\langle f, P\mu\rangle = \langle P^*f, \mu\rangle \tag{5.13}$$

for every bounded Borel function f and every finite Borel measure $\mu \in \mathcal{M}_+$. To see this, pick any $B \in \mathcal{B}$. Evidently (5.13) holds when $f = \mathbf{1}_B$. By linearity of the inner product, (5.13) also holds when f is a step function taking only finitely many values. This can be extended from step functions to any bounded nonnegative Borel function by pointwise approximation and a monotone convergence result in the usual way. Linearity then implies the result for an arbitrary bounded Borel function f, which can always be written as the difference between two nonnegative parts.

Assume now that φ satisfies (5.12). Then

$$\langle P^*f, \varphi\rangle = \langle f, \varphi\rangle, \quad \forall f \in C_b.$$

Therefore,

$$\langle P^* f, \varphi \rangle = \langle f, \varphi \rangle, \quad \forall f \in C_0. \tag{5.14}$$

Since φ is a finite Borel measure and since each $f \in C_0$ is a bounded Borel function, together (5.13) and (5.14) imply that

$$\langle f, P\varphi \rangle = \langle f, \varphi \rangle, \quad \forall f \in C_0. \tag{5.15}$$

This says precisely that the positive linear functionals on C_0 generated by the two measures $P\varphi$ and φ in the manner of (5.11) are identical. Given that $P\varphi$ and φ are finite Borel measures, and that the association (5.11) from $\mathcal{M}_+$ to C_0^* is one-to-one, this implies that the representing measures $P\varphi$ and φ are identical. This is equivalent to stating that φ satisfies (5.2).

The converse is obvious, and completes the proof of the lemma. $\square$

Chapter 6

Linearization of Stochastic Economic Models

6.1 Introduction

Consider again a stochastic macroeconomic system with the stylized representation

$$x_{t+1} = T(x_t, \varepsilon_t), \quad t = 0, 1, \ldots, \tag{6.1}$$

where x is a collection of endogenous variables taking values in topological vector space X, T is an arbitrary function with $\operatorname{rng} T \subset X$, and (ε_t) is a sequence of serially uncorrelated random variables. Given (6.1), the researcher seeks to characterize dynamics in terms of the sequence (x_t). Ideally, the distribution of x_t will converge to a unique limiting distribution independent of x_0 as $t \to \infty$. As usual, this distribution is defined to be the equilibrium of the economy, and is a focal point for policy simulation and other analysis.

In the applied literature, a common approach to the analysis of stochastic dynamics is via linearization. There are two standard methods. One is to replace ε_t in (6.1) with its mean, solve the resulting deterministic model for fixed points, and linearize in their vicinity by Taylor expansion. (For an exposition, see, e.g., Farmer [18, Sections 2.3.3 and 2.3.4].) The other method is log-linearization, which is of course only applicable when the underlying model is log-linear. (See, e.g., Long and Plosser [42].)

However, it must be recalled at all times that the linearized system is *auxiliary* to the analysis: it is valuable only to the extent that it provides insight into the dynamic properties of the true model (6.1). In this connection, we stress that for stochastic systems such as (6.1), it is *not* in general legitimate to infer such properties as existence, uniqueness and stability of equilibrium from similar properties as they may or may not occur in the linear version.

In this final essay we begin to address this issue by giving a formal justification for log-linearization of stochastic models. It is shown that for this particular case parallel existence, uniqueness and stability results hold for equilibria in the original and linearized models.

Section 6.2 formulates the problem. Section 6.3 states results. Section 6.4 gives an application to the multisector macroeconomic model of Long and Plosser. Section 6.5 gives proofs.

6.2 Formulation of the Problem

In this section we briefly recall the definitions of equilibria and stability for the model (6.1). Let topological space X be the state space for (6.1). That is, $x_t \in X$ for all t, and, assuming that ε_t also takes values in X, the map T satisfies $T\colon X \times X \to X$. Let $\mathcal{B} = \mathcal{B}(X)$ be the Borel subsets of X, and let $\mathcal{P} = \mathcal{P}(X)$ be the set of probabilistic measures mapping events $B \in \mathcal{B}$ into probabilities in $[0, 1]$. In other words, $\mathcal{P}$ is the class of countably additive, real valued functions ν on $\mathcal{B}$ such that $\nu \geq 0$ and $\nu(X) = 1$. As before, $\mathcal{P}$ is metrized by the total variation norm.

The shocks ε_t in (6.1) are all drawn independently according to some fixed distribution $\Psi \in \mathcal{P}$. Given an initial condition x_0, a shock ε_0 is drawn, and x_1 is realized according to (6.1). The process then repeats. Evidently x_t is an X-valued random variable. We denote the distribution of x_t by $\nu_t \in \mathcal{P}$.

We use again the following conventions. The symbol $\mathbf{1}_E$ denotes the characteristic function of $E \subset X$. The notation $x \sim \nu$ means that random variable x has distribution ν; for function g, the notation g^n means the n-th composition of g with itself.

By Definition 2.3, page 28, an *equilibrium* for (6.1) is a distribution $\nu_* \in \mathcal{P}$ such that

$$\nu_*(B) = \int \left[\int \mathbf{1}_B[T(x,z)]\Psi(dz) \right] \nu_*(dx), \quad \forall\, B \in \mathcal{B}. \tag{6.2}$$

The equilibrium ν_* is unique if there exists no other measure in $\mathcal{P}$ satisfying (6.2). The equilibrium is called *globally stable* if $\nu_t \to \nu_*$ in the norm topology as $t \to \infty$ for every initial condition ν_0 (i.e., $x_0 \sim \nu_0$).

6.3 Results

Consider now the case where (6.1) is log-linear. Let $\mathbb{R}_{++} = (0, \infty)$. The general form of a stochastic log-linear (log-affine) system on $\mathbb{R}^n_{++} = \times_{i=1}^n \mathbb{R}_{++}$ is

$$\begin{aligned} x_{1,t+1} &= \gamma_1 x_{1t}^{a_{11}} \times \cdots \times x_{nt}^{a_{1n}} \varepsilon_{1t} \\ &\vdots \qquad\qquad \vdots \\ x_{n,t+1} &= \gamma_n x_{1t}^{a_{n1}} \times \cdots \times x_{nt}^{a_{nn}} \varepsilon_{nt}. \end{aligned} \tag{NL}$$

Here $\gamma = (\gamma_i)_{i=1}^n \in \mathbb{R}^n_{++}$, and the vector of shocks $\varepsilon_t = (\varepsilon_{it})_{i=1}^n \in \mathbb{R}^n_{++}$ is assumed to be serially independent and distributed by $\Psi \in \mathcal{P}(\mathbb{R}^n_{++})$.

Given n-dimensional vector x, it is convenient to use the abbreviations $\ln x$ for $(\ln x_i)_{i=1}^n$ and $\exp x$ for $(\exp x_i)_{i=1}^n$. We also use $\ln B$ to denote the set of points $\ln x$, $x \in B$, and $\exp B$ for all points $\exp x$, $x \in B$. For example, $\ln \mathbb{R}^n_{++} = \mathbb{R}^n$.

After taking logs, the linear version of (NL) is

$$x_{t+1} = \hat{\gamma} + A x_t + \hat{\varepsilon}_t, \quad x \in \mathbb{R}^n, \tag{LV}$$

where $\hat{\gamma} = \ln \gamma$, $\hat{\varepsilon}_t = \ln \varepsilon_t$ and A is the $n \times n$ matrix (a_{ij}).

Our main result is as follows.

Proposition 6.1. *The log-linear economy* (NL) *has a unique equilibrium* $\nu_* \in \mathcal{P}(\mathbb{R}^n_{++})$ *if and only if the linear version* (LV) *has a unique equilibrium* $\hat{\nu}_* \in \mathcal{P}(\mathbb{R}^n)$. *The equilibrium* ν_* *for* (NL) *can be recovered from* $\hat{\nu}_*$ *by the identity* $\nu_*(B) = \hat{\nu}_*(\ln B)$, $\forall B \in \mathcal{B}(\mathbb{R}^n_{++})$. *The equilibrium* ν_* *is globally stable if and only if* $\hat{\nu}_*$ *is globally stable.*

The proof is given in Section 6.5. The argument is based on establishing topological conjugacy between the two systems.

6.4 Application

A well-know study using log-linearization is that of Long and Plosser [42, pp. 52–54]. As an application of Proposition 6.1, in this section we complete their analysis by verifying that the stability of the linearized system that they discuss does in fact (under suitable conditions) imply stability of the original model.

The model is an infinite horizon, representative agent economy with n sectors. Let $c_t = (c_{it})_{i=1}^n$ be time t consumption. Utility is given by $u(c_t) = \sum_{i=1}^n \theta_i \ln c_{it}$, $\theta_i > 0$. Production is according to the Cobb-Douglas technology

$$y_{i,t+1} = \ell_{it}^{b_i} x_{i1t}^{a_{i1}} \times \cdots \times x_{int}^{a_{in}} \varepsilon_{it}, \quad i = 1, \ldots n, \tag{6.3}$$

where y_i is output of commodity i, ℓ_i is labor allocated to sector i, x_{ij} is the amount of commodity j used in the production of good i, and ε_i is a sector-specific shock. The vector of shocks is uncorrelated and identically distributed over time.[1]

Production is assumed to be constant returns to scale. In particular,

$$b_i, a_{ij} > 0; \quad b_i + \sum_{j=1}^n a_{ij} = 1, \quad i = 1, \ldots, n. \tag{6.4}$$

[1] The objective of Long and Plosser was to generate fluctuations in time series consistent with the business cycle from a general equilibrium framework and without serial dependence in external noise.

The economy faces constraints

$$c_{jt} + \sum_{i=1}^{n} x_{ijt} \leq y_{jt}, \quad j = 1, \ldots, n, \tag{6.5a}$$

$$\sum_{i=1}^{n} \ell_i \leq L, \quad i = 1, \ldots, n. \tag{6.5b}$$

The representative agent seeks a solution to

$$\max \mathbb{E} \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right]$$

subject to (6.3) and (6.5), where $\beta \in (0, 1)$ is a discount factor. The optimal controls are

$$x_{ijt} = a_{ij} \beta \frac{\lambda_i}{\lambda_j} y_{jt}, \tag{6.6a}$$

$$\ell_{it} = \lambda_i b_i \Big(\sum_{j=1}^{n} \lambda_j b_j \Big)^{-1} L, \tag{6.6b}$$

where $\lambda' = \theta'(1 - \beta A)$, A being the matrix (a_{ij}) of output elasticities with respect to commodity inputs [42, pp. 47–48]. Substitution of (6.6) into (6.3) gives

$$\begin{aligned} y_{1,t+1} &= \gamma_1 y_{1t}^{a_{11}} \times \cdots \times y_{nt}^{a_{1n}} \varepsilon_{1t} \\ \vdots & \qquad\qquad \vdots \\ y_{n,t+1} &= \gamma_n y_{1t}^{a_{n1}} \times \cdots \times y_{nt}^{a_{nn}} \varepsilon_{nt}, \end{aligned} \tag{LP}$$

where $\gamma = (\gamma_i)_{i=1}^n$ is a vector of positive constants.

Following Long and Plosser, we can convert (LP) into a linear form and observe global stability of the latter. By virtue of Proposition 6.1, this implies conditions under which (LP) is itself globally stable. Formally,

Proposition 6.2. *Let ε be the vector of sectoral shocks in* (LP)*. If, for some norm $\|\cdot\|$ on $\mathbb{R}^n$, the expectation $\mathbb{E}\,\|\ln\varepsilon\|$ is finite, then the economy* (LP) *has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n_{++})$.*

Proof. The linear version of (LP) is

$$y_{t+1} = \hat{\gamma} + Ay_t + \hat{\varepsilon}_t, \quad y \in \mathbb{R}^n, \tag{6.7}$$

where, as before, $\hat{\gamma} = \ln\gamma$, $\hat{\varepsilon} = \ln\varepsilon$, and $A = (a_{ij})$ is the matrix of input/output elasticity coefficients. The linear stochastic system (6.7) is well understood. In particular, it is known that if, for some norm $\|\cdot\|$ on $\mathbb{R}^n$, $\mathbb{E}\,\|\hat{\varepsilon}\|$ is finite, and, in addition, that the spectral radius of A is less than one, then (6.7) has a unique and globally stable equilibrium distribution $\hat{\nu}_* \in \mathcal{P}(\mathbb{R}^n)$.[2]

The spectral radius of any nonnegative matrix is less than or equal to the maximum of the row sums [37, Theorem 7.2.1]. In the case of A, these sums are all strictly less than one by (6.4). It follows that, under the hypothesis $\mathbb{E}\,\|\hat{\varepsilon}\| < \infty$, (6.7) has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n)$.

But then $\mathbb{E}\,\|\ln\varepsilon\| < \infty$ implies that (LP) has a unique, globally stable equilibrium in $\mathcal{P}(\mathbb{R}^n_{++})$ by Proposition 6.1. □

Remark 6.1. The hypotheses of Proposition 6.2 are satisfied if, for example, $\mathbb{E}\,|\ln\varepsilon_i|$ is finite for each i. This condition enforces small left- and right-hand tails on the distributions of the sectoral shocks ε_i. These small tails prevent the economy from either collapsing to zero or growing without bound.

[2]See, for example, Lasota and Mackey [40, Proposition 12.7.1, Theorem 12.7.2].

6.5 Proofs

It remains to prove Proposition 6.1. The method is as follows. First, we recall the notion of topological conjugacy between dynamical systems; conjugate systems have identical dynamics. Since the notion of conjugacy is defined for deterministic rather than stochastic systems, our next step is to convert the log-linear and linear systems (NL) and (LV) into deterministic self-mappings on spaces of probability measures. Finally, we show that these deterministic versions are topologically conjugate.

Recall that by a *homeomorphism* is meant a bijection from one topological space to another that is continuous and has continuous inverse. Let X and $\hat{X}$ be two metrizable spaces. Consider the two dynamical systems

$$x_{t+1} = g(x_t), \quad g \colon X \to X, \tag{6.8}$$

$$\hat{x}_{t+1} = \hat{g}(\hat{x}_t), \quad \hat{g} \colon \hat{X} \to \hat{X}. \tag{6.9}$$

Suppose that, corresponding to g and $\hat{g}$, there exists a homeomorphism H from X into $\hat{X}$ such that g and $\hat{g}$ commute with H in the sense that $\hat{g} = HgH^{-1}$ on X. Then (6.8) and (6.9) are said to be *topologically conjugate.*

In this case, (6.8) has a unique, globally stable equilibrium (i.e., fixed point x_* of g on X such that $\lim g^t(x) \to x_*$ as $t \to \infty$, $\forall x \in X$) if and only if (6.9) has a unique, globally stable equilibrium. These results are well-known [3, Section 3.3] and not difficult to verify. For example, if x_* is a fixed point of g on X, then $\hat{x}_* = Hx_*$ is a fixed point of $\hat{g}$ on $\hat{X}$, because $g(x_*) = x_*$, and therefore $\hat{g}(\hat{x}_*) = HgH^{-1}Hx_* = Hg(x_*) = Hx_* = \hat{x}_*$.

Thus to prove Proposition 6.1, it remains only to rewrite the nonlinear and linear systems (NL) and (LV) in the form of (6.8) and (6.9), and show that they commute with a suitable homeomorphism.

To rewrite these systems in the deterministic form of (6.8) and (6.9), we use the techniques of Chapter 2, Section 2.3. To aid the exposition these arguments are presented here again, albeit rather tersely.

Consider again the generic system (T, Ψ) on X discussed in Section 6.2. Let $\nu_t \in \mathcal{P}(X)$ be the marginal distribution of the random variable x_t, and let ν_{t+1} be that of x_{t+1}. Then ν_t and ν_{t+1} are connected by the recursion

$$\nu_{t+1}(B) = \int \left[\int \mathbf{1}_B[T(x, z)]\Psi(dz) \right] \nu_t(dx), \quad \forall\, B \in \mathcal{B}(X). \tag{6.10}$$

To repeat, the intuition is that the right hand side of (6.10) sums the probability of the state moving to B from x in one step over all possible values of x, weighted by the probability $\nu_t(dx)$ of x occurring as the current state.

Let Q be the associated Markov operator from $\mathcal{P}(X)$ into itself:

$$(Q\nu)(B) = \int \left[\int \mathbf{1}_B[T(x, z)]\Psi(dz) \right] \nu(dx), \quad B \in \mathcal{B}(X).$$

In this form, Q is sometimes called the *Foias operator* corresponding to (6.1). Using Q allows (6.10) to be rewritten as $\nu_{t+1} = Q\nu_t$. By recursion, if ν_0 is the initial state for the system (6.1) in the sense that $x_0 \sim \nu_0$, then $Q^t\nu_0$ is the distribution for the state at time t, where Q^t is defined by $Q^t = QQ^{t-1}$, $Q^1 = Q$. In light of (6.2), an equilibrium for the system (6.1) is a distribution $\nu_* \in \mathcal{P}$ such that $Q\nu_* = \nu_*$. The equilibrium is globally stable if and only if $Q^t\nu_0 \to \nu_*$ as $t \to \infty$ for all $\nu_0 \in \mathcal{P}(X)$ (see Definition 2.3, p. 28).

Denote by P and $\hat{P}$ the Foias operators associated with the log-linear system (NL) and the linear version (LV) respectively. These two systems can now be represented as

$$\nu_{t+1} = P\nu_t, \quad P\colon \mathcal{P}(\mathbb{R}^n_{++}) \to \mathcal{P}(\mathbb{R}^n_{++}), \tag{6.11}$$

$$\hat{\nu}_{t+1} = \hat{P}\hat{\nu}_t, \quad \hat{P}\colon \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n). \tag{6.12}$$

Note that the pair (6.11) and (6.12) are in the same deterministic form as (6.8) and (6.9). Thus to complete the proof of Proposition 6.1 we need to establish a homeomorphism H from $\mathcal{P}(\mathbb{R}^n_{++})$ onto $\mathcal{P}(\mathbb{R}^n)$ such that P and $\hat{P}$ commute with H, in the sense that $P = H^{-1}\hat{P}H$ on $\mathcal{P}(\mathbb{R}^n_{++})$.

A suitable candidate for a homeomorphism is the map $H\colon \mathcal{P}(\mathbb{R}^n_{++}) \to \mathcal{P}(\mathbb{R}^n)$ defined at $\nu \in \mathcal{P}(\mathbb{R}^n_{++})$ by

$$(H\nu)(B) = \nu(\exp B), \quad B \in \mathcal{B}(\mathbb{R}^n). \tag{6.13}$$

It is not difficult to verify that H is a one-to-one correspondence from $\mathcal{P}(\mathbb{R}^n_{++})$ onto $\mathcal{P}(\mathbb{R}^n)$, where if $\hat{\nu} \in \mathcal{P}(\mathbb{R}^n)$, then $H^{-1}\hat{\nu}(B) = \hat{\nu}(\ln B)$.

Thus H is a bijection. In fact,

Lemma 6.1. *The map H defined in* (6.13) *is a homeomorphism.*

Proof. Regarding continuity of H, let $\nu_n \to \nu$ in $\mathcal{P}(\mathbb{R}^n_{++})$. It is shown in Stokey et al. [58, Theorem 11.6] that, for any metric space X, any sequence (ν_n) in $\mathcal{P}(X)$ and any $\nu \in \mathcal{P}(X)$, the statement $\nu_n \to \nu$ in total variation norm is equivalent to $|\nu_n(B) - \nu(B)| \to 0$ uniformly on $\mathcal{B}(X)$.

Fix $\varepsilon > 0$. By hypothesis, there exists an $N \in \mathbb{N}$ such that $n \geq N$ implies

$$|\nu_n(A) - \nu(A)| < \varepsilon, \quad \forall A \in \mathcal{B}(\mathbb{R}^n_{++}).$$

But then

$$n \geq N \implies |(H\nu_n)(B) - (H\nu)(B)| = |\nu_n(\exp B) - \nu(\exp B)| < \varepsilon$$

for any $B \in \mathcal{B}(\mathbb{R}^n)$. Hence $H\nu_n \to H\nu$ in $\mathcal{P}(\mathbb{R}^n)$. This proves continuity of H. The proof of continuity of H^{-1} is similar. □

To complete the proof of Proposition 6.1 we need show only that

Lemma 6.2. *The relation $P = H^{-1}\hat{P}H$ holds on $\mathcal{P}(\mathbb{R}^n_{++})$.*

Proof. Prior to the main proof we briefly recall how to integrate with respect to the induced measure $H\nu$ [1, Theorem 12.46].

Fix $\nu \in \mathcal{P}(\mathbb{R}^n_{++})$. If $h\colon \mathbb{R}^n \to \mathbb{R}$ is any $\mathcal{B}(\mathbb{R}^n)$-measurable function that is summable with respect to $H\nu \in \mathcal{P}(\mathbb{R}^n)$, then $\mathbb{R}^n_{++} \ni x \mapsto h(\ln x) \in \mathbb{R}$ is $\mathcal{B}(\mathbb{R}^n_{++})$-measurable, and

$$\int_{\mathbb{R}^n} h(x)(H\nu)(dx) = \int_{\mathbb{R}^n_{++}} h(\ln x)\nu(dx). \tag{6.14}$$

Also, note that the system (NL) can be expressed more compactly as

$$x_{t+1} = \exp(\hat{c} + A\ln x_t + \ln \varepsilon_t), \quad x \in \mathbb{R}^n_{++}, \tag{6.15}$$

We now show that

$$(\hat{P}H\nu)(B) = (HP\nu)(B), \quad \forall \nu \in \mathcal{P}(\mathbb{R}^n_{++}), \quad \forall B \in \mathcal{B}(\mathbb{R}^n), \tag{6.16}$$

which is equivalent to the statement of the lemma. Note first that $\varepsilon \sim \Psi \in \mathcal{P}(\mathbb{R}^n_{++})$ implies $\hat{\varepsilon} \sim H\Psi \in \mathcal{P}(\mathbb{R}^n)$, because for all $B \in \mathcal{P}(\mathbb{R}^n)$, $\text{Prob}[\hat{\varepsilon} \in B] = \text{Prob}[\ln \varepsilon \in B] = \text{Prob}[\varepsilon \in \exp B] = \Psi(\exp B) = H\Psi(B)$.

Hence

$$\begin{aligned}
(\hat{P}H\nu)(B) &= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} \mathbf{1}_B(\hat{c} + Ax + z)(H\Psi)(dz) \right] H\nu(dx) \\
&= \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n_{++}} \mathbf{1}_B(\hat{c} + Ax + \ln z)\Psi(dz) \right] H\nu(dx) \\
&= \int_{\mathbb{R}^n_{++}} \left[\int_{\mathbb{R}^n_{++}} \mathbf{1}_B(\hat{c} + A\ln x + \ln z)\Psi(dz) \right] \nu(dx) \\
&= \int_{\mathbb{R}^n_{++}} \left[\int_{\mathbb{R}^n_{++}} \mathbf{1}_{\exp B}[\exp(\hat{c} + A\ln x + \ln z)]\Psi(dz) \right] \nu(dx),
\end{aligned}$$

where we have used (6.14) to change variables.

But the representation (6.15) shows that this is just $HP\nu(B)$, which proves (6.16). $\square$

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Appendix A

Existing Conditions

We now state and prove the problem of Chapter 4, Section 4.3.3.

Let (g, ψ) be a perturbed dynamical system on $\mathbb{R}_+$ satisfying Assumptions 4.1 and 4.2. Horbacz [30, Theorem 1] shows that (g, ψ) has a unique and globally stable equilibrium whenever

(i) The map g is weakly monotone increasing and continuously differentiable on $[0, r) \neq \varnothing$, and $g(x) \geq b > 0$ on $[r, \infty)$;

(ii) the map g satisfies $g(0) = 0$ and $g'(0) > 0$;

(iii) there exist a, $B \geq 0$ such that $g(x) \leq ax + B$ for all $x \in \mathbb{R}_+$;

(iv) the mean $\mathbb{E}(\varepsilon) = \int z\psi(z)dz$ is finite and, moreover, $\mathbb{E}(\varepsilon)a < 1$;

(v) there exists a $\lambda > 0$ such that $\mathbb{E}[(g'(0)\varepsilon)^{-\lambda}] < 1$; and

(vi) the density ψ is everywhere positive on $\mathbb{R}_+$.

Proposition A.1. *Conditions* (i)–(vi) *of Horbacz are a special case of Theorem 4.2.*

Proof. Evidently Conditions 4.2 and 4.4 of the theorem are satisfied. It remains to verify Condition 4.1. To this end, let λ be as in (v). If we set $V(0) = \infty$ and $V(x) = x^{-\lambda} + x$ for $x > 0$, then V is a Lyapunov function on $\mathbb{R}_+$, and

$$\int V[g(x)z]\psi(z)dz = \int [g(x)z]^{-\lambda}\psi(z)dz + \int g(x)z\psi(z)dz. \tag{A.1}$$

Consider the first term in the sum (A.1). By (v), there exists a positive number σ so small that

$$\int [(g'(0) - \sigma)z]^{-\lambda}\psi(z)dz < 1. \tag{A.2}$$

By (i) and (ii), there exists a $\delta > 0$ such that

$$g(x) \geq (g'(0) - \sigma)x \quad \text{whenever } x \in [0, \delta). \tag{A.3}$$

Combining (A.2) and (A.3) yields a $\gamma < 1$ such that

$$\int [g(x)z]^{-\lambda}\psi(z)dz \leq \gamma x^{-\lambda}, \quad \forall\, x \in [0, \delta).$$

Moreover, (i) implies the existence of a $c > 0$ such that

$$g(x) \geq c \quad \text{whenever } x \in [\delta, \infty).$$

Thus, for all $x \in \mathbb{R}_+$, we have the bound

$$\int [g(x)z]^{-\lambda}\psi(z)dz \leq \gamma x^{-\lambda} + C_0, \tag{A.4}$$

where $\gamma < 1$ and C_0 is a finite constant.

Regarding the second term in the sum (A.1), (iii) implies that

$$\int g(x)z\psi(z)dz \leq \mathbb{E}(\varepsilon)ax + C_1, \quad x \in \mathbb{R}_+, \tag{A.5}$$

where C_1 is a finite constant.

Combining (A.4) and (A.5) gives

$$\int V[g(x)z]\psi(z)dz \leq \alpha V(x) + C, \tag{A.6}$$

where $\alpha = \max[\mathbb{E}(\varepsilon)a, \gamma] < 1$ and $C = C_0 + C_1 < \infty$. This confirms Condition 4.1. Hence all of the conditions of Theorem 4.2 are satisfied. $\square$