ECON-GA 1025 Macroeconomic Theory I

Lecture 7

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Today's Lecture

- VARs and linear state space processes
- Random coefficient models
- Nonlinear stochastic models
- Numerical methods for tracking distributions

Distribution Dynamics: The General Density Case

Recall $x_{t+1} = Ax_t + b + C\xi_{t+1}$

Assume now

- $\{\xi_t\}$ is IID on $\mathbb{R}^n$ with density φ
- C is $n \times n$ and nonsingular

Under these assumptions, each ψ_t will be a density

To prove this we use

Fact. If ξ has density φ and C is nonsingular, then $y = d + C\xi$ has density

$$p(y) = \varphi\left(C^{-1}(y - d)\right) |\det C|^{-1}$$

The density of x_{t+1} conditional on $x_t = x$ is therefore

$$\pi(x,y) = \varphi\left(C^{-1}(y - Ax - b)\right) |\det C|^{-1}$$

The law of total probability tells us that, for random varables (x,y) with densities,

$$p(y) = \int p(y \,|\, x) p(x) \, \mathrm{d}x$$

Hence the densities ψ_t and ψ_{t+1} are connected via

$$\psi_{t+1}(y) = |\det C|^{-1} \int \varphi\left(C^{-1}(y - Ax - b)\right) \psi_t(x) \, \mathrm{d}x$$

Suppose we introduce an operator Π from the set of densities $\mathcal{D}$ on $\mathbb{R}^n$ to itself via

$$(\psi\Pi)(y) = \int \pi(x,y)\psi(x)\,\mathrm{d}x$$

Then our law of motion for marginals

$$\psi_{t+1}(y) = |\det C|^{-1} \int \varphi\left(C^{-1}(y - Ax - b)\right) \psi_t(x)\,\mathrm{d}x$$

becomes

$$\psi_{t+1} = \psi_t \Pi$$

- a concise description of distribution dynamics

Comments:

- In $\psi_{t+1} = \psi_t \Pi$ we write the argument to the left following tradition (see Meyn and Tweedie, 2009)
- The set of densities $\mathcal{D}$ is endowed with the topology of weak convergence

Proposition. If $r(A) < 1$, then $(\mathcal{D}, \Pi)$ is globally stable

Moreover, if h is any function such that $\int |h(x)| \psi^*(x) \, \mathrm{d}x$ is finite, then

$$\mathbb{P} \left\{ \lim_{n \to \infty} \frac{1}{n} \sum_{t=1}^{n} h(x_t) = \int h(x) \psi^*(x) \, \mathrm{d}x \right\} = 1$$

Linear State Space Models

The standard **linear state space** model is

$$x_{t+1} = Ax_t + b + C\xi_{t+1}$$

$$y_t = Gx_t + H\zeta_t$$

where

- A is $n \times n$, b is $n \times 1$ and C is $n \times j$
- G is $k \times n$ and H is $k \times \ell$
- $\{\xi_t\}$ are IID $j \times 1$ with $\mathbb{E}\xi_t = 0$ and $\mathbb{E}\xi_t\xi_t' = I$
- $\{\zeta_t\}$ are IID $\ell \times 1$ with $\mathbb{E}\zeta_t = 0$ and $\mathbb{E}\zeta_t\zeta_t' = I$

In this context

- $\{x_t\}$ is called the **state process**
- $\{y_t\}$ is called the **observation process**

Example. The standard linear model of log labor earnings discussed in is

$$y_t = x_t + h\zeta_t \quad \text{where} \quad x_{t+1} = \rho x_t + b + c\xi_{t+1}$$

- $\{\xi_t\}$ and $\{\zeta_t\}$ are IID and standard normal in $\mathbb{R}$
- h, ρ, b, c are parameters, with $|\rho| < 1$

Recalling that

- $\mu_t = g^t(\mu_0)$ where $g(\mu) := A\mu + b$
- $\Sigma_t = S^t(\Sigma_0)$ where $S(\Sigma) := A'\Sigma A + CC'$

we obtain

$$\mathbb{E} y_t = G\mu_t \quad \text{and} \quad \operatorname{var} y_t = G\Sigma_t G' + HH'$$

If $r(A) < 1$, then

$$\mathbb{E} y_t \to G\mu^*, \quad \text{and} \quad \operatorname{var} y_t \to G\Sigma^* G' + HH'$$

where $\mu^*, \Sigma^* =$ fixed points of g, S

Ergodicity results also hold when $r(A) < 1$

For example,

$$\begin{aligned}
\frac{1}{n} \sum_{t=1}^{n} y_t &= \frac{1}{n} \sum_{t=1}^{n} (G x_t + H \zeta_t) \\
&= G \frac{1}{n} \sum_{t=1}^{n} x_t + H \frac{1}{n} \sum_{t=1}^{n} \zeta_t \\
&\to G \mu^*
\end{aligned}$$

with prob one as $n \to \infty$

Forecasts

We wish to forecast geometric sums

Example. If $\{y_t\}$ is a cash flow, what is the expected discounted value?

The formulas are

$$\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j x_{t+j} = [I - \beta A]^{-1} x_t,$$

$$\mathbb{E}_t \sum_{j=0}^{\infty} \beta^j y_{t+j} = G[I - \beta A]^{-1} x_t$$

Ex. Show the formulas are valid whenever $r(A) < 1/\beta$

Nonlinear Stochastic Models

We have looked at

1. nonlinear deterministic models
2. nonlinear stochastic models on discrete state spaces and
3. linear stochastic models

Now we turn to general nonlinear stochastic models on continuous state spaces

First some motivation...

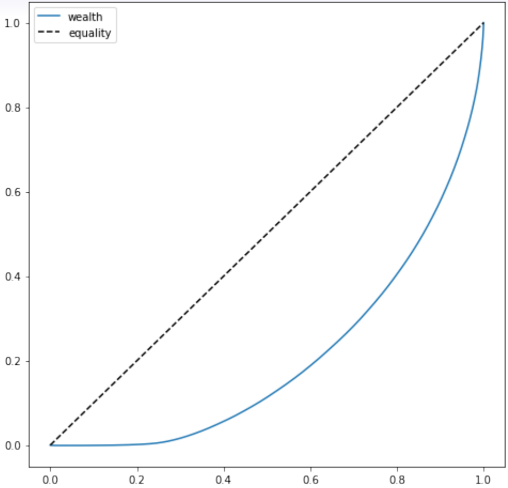

Figure: Lorenz curve, wealth distribution in Italy

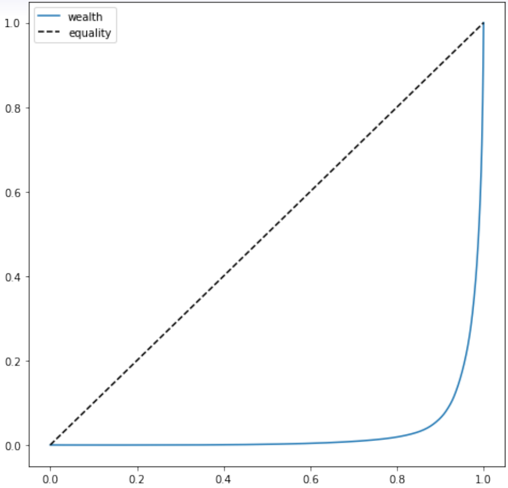

Figure: Lorenz curve, wealth distribution in the US (SCF 2016)

Consider a **first order Markov process** on **state space** $\mathsf{X} \subset \mathbb{R}^k$ defined by

$$X_{t+1} = F(X_t, \xi_{t+1})$$

where

- $\{\xi_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} \Phi$ in $E \subset \mathbb{R}^j$
- $F \colon \mathsf{X} \times E \to \mathsf{X}$ is Borel measurable

Notes:

- the shock distribution Φ is a CDF
- The initial condition is X_0 with CDF Ψ_0

Assume: X_0 is independent of process $\{\xi_t\}$

Implies independence of X_t and ξ_{t+1} for all t

This holds because X_t is a function only of X_0 and $\xi_1, \ldots, \xi_t$

$$X_1 = F(X_0, \xi_1)$$

$$X_2 = F(F(X_0, \xi_1), \xi_2)$$

$$X_3 = F(F(F(X_0, \xi_1), \xi_2), \xi_3)$$

and so on

Example. Consider a stochastic Solow–Swan model on $(0, \infty)$ where

$$k_{t+1} = s z_{t+1} f(k_t) + (1 - \delta) k_t \quad \text{where} \quad \{z_t\} \overset{\text{IID}}{\sim} \varphi \text{ on } (0, \infty)$$

A first order Markov process with

- state variable k_t taking values in $\mathsf{X} = (0, \infty)$,
- shock space $E = (0, \infty)$ and
- law of motion $F(k, z) := s z f(k) + (1 - \delta) k$,

Example. Consider stochastic Solow–Swan model growth model

- $k_{t+1} = sz_{t+1}f(k_t) + (1-\delta)k_t$
- $z_t = \exp(y_t)$ where $y_{t+1} = ay_t + b + c\xi_{t+1}$
- $\{\xi_t\}$ IID and $N(0,1)$

A first order Markov process with

- state vector $X_t := (k_t, y_t) \in \mathsf{X} := (0,\infty) \times \mathbb{R}$,
- law of motion

$$F((k,y),\xi) = \begin{pmatrix} s\exp(ay+b+c\xi)f(k) + (1-\delta)k \\ ay+b+c\xi \end{pmatrix}$$

Let Ψ_t represent the CDF of the state vector X_t generated by

$$X_{t+1} = F(X_t, \xi_{t+1}), \qquad \{\xi_t\}_{t \geqslant 1} \overset{\text{IID}}{\sim} \Phi$$

By independence of X_t and ξ_{t+1},

$$\begin{aligned} \mathbb{P}\{X_{t+1} \leqslant y\} &= \mathbb{E}\mathbb{1}\{F(X_t, \xi_{t+1}) \leqslant y\} \\ &= \int \int \mathbb{1}\{F(x, z) \leqslant y\} \Phi(\mathrm{d}z) \Psi_t(\mathrm{d}x) \end{aligned}$$

(The joint CDF of (X_t, ξ_{t+1}) is just the product of the marginals)

The last equation can be written as

$$\Psi_{t+1}(y) = \int \int \mathbb{1}\{F(x,z) \leqslant y\} \Phi(\mathrm{d}z) \Psi_t(\mathrm{d}x)$$

Alternatively,

$$\Psi_{t+1}(y) = \int \Pi(x,y) \Psi_t(\mathrm{d}x) \qquad (y \in \mathsf{X})$$

where

$$\Pi(x,y) := \int \mathbb{1}\{F(x,z) \leqslant y\} \Phi(\mathrm{d}z)$$

$=:$ the **stochastic kernel** for our model

We can write

$$\Psi_{t+1}(y) = \int \Pi(x, y) \Psi_t(\mathrm{d}x) \qquad (y \in \mathsf{X})$$

as

$$\Psi_{t+1} = \Psi_t \Pi$$

Where Π is the operator on CDF space defined by

$$(\Psi \Pi)(y) = \int \Pi(x, y) \Psi(\mathrm{d}x) \qquad (y \in \mathsf{X})$$

If $\mathcal{P}(\mathsf{X})$ is the set of all distributions on X, then

- $(\mathcal{P}(\mathsf{X}), \Pi)$ forms a dynamical system
- a **stationary distribution** is a fixed point of Π in $\mathcal{P}(\mathsf{X})$

Example. The marginal distributions $\{\Psi_t\}$ of capital under the Solow–Swan model obey

$$\Psi_{t+1}(k') = \int \Pi(k, k') \Psi_t(\mathrm{d}k) \qquad (k > 0)$$

with

$$\begin{aligned}\Pi(k, k') &= \mathbb{P}\{s\xi_{t+1} f(k) + (1-\delta)k \leqslant k'\} \\ &= \Phi\left(\frac{k' - (1-\delta)k}{sf(k)}\right)\end{aligned}$$

A distribution Ψ^* is stationary if

$$\Psi^*(k') = \int \Phi\left(\frac{k' - (1-\delta)k}{sf(k)}\right) \Psi^*(\mathrm{d}k) \qquad (k > 0)$$

The Density Case

The sequence $\{\Psi_t\}$ has density representations $\{\psi_t\}$ in some cases

Key condition: $\Pi(x, \cdot)$ can be represented by a density $\pi(x, \cdot)$

Formally, exists for each $x \in \mathsf{X}$ a $\pi(x, \cdot)$ such that

$$\Pi(x, y) = \int_{u \leqslant y} \pi(x, u) \, \mathrm{d}u$$

- π is called a **density stochastic kernel**

$\Longrightarrow$ distributions $\{\Psi_t\}$ all have densities $\{\psi_t\}$ and they satisfy

$$\psi_{t+1}(y) = \int \pi(x, y) \psi_t(x) \, \mathrm{d}x$$

Example. Consider the IID Solow–Swan CDF kernel

$$\Pi(k, k') = \Phi\left(\frac{k' - (1-\delta)k}{sf(k)}\right)$$

If Φ is differentiable, $\Phi' = \varphi$, then, differentiating w.r.t. k',

$$\pi(k, k') = \varphi\left(\frac{k' - (1-\delta)k}{sf(k)}\right)\frac{1}{sf(k)}$$

The marginal densities $\{\psi_t\}$ satisfy

$$\psi_{t+1}(k') = \int \varphi\left(\frac{k' - (1-\delta)k}{sf(k)}\right)\frac{1}{sf(k)}\psi_t(k)\,\mathrm{d}k$$

Random Coefficient Models

Kesten processes or **random coefficient models** are recursive sequences of the form

$$x_{t+1} = A_{t+1} x_t + \eta_{t+1}$$

where

- $\{x_t\}_{t \geqslant 0}$ is an $n \times 1$ state vector process
- $\{A_t\}$ is IID, takes values in $\mathcal{M}(n \times n)$
- $\{\eta_t\}$ is IID, takes values in $\mathbb{R}^n$

Stochastic kernel, is, in CDF format,

$$\Pi(x, y) = \mathbb{P}\{A_{t+1} x + \eta_{t+1} \leqslant y\} \qquad (y \in \mathbb{R}^n)$$

Assume:

$$\mathbb{E}\|A_t\| < \infty \quad \text{and} \quad \mathbb{E}\|\eta_t\| < \infty$$

Let

$$L_n := \frac{1}{n}\,\mathbb{E}\ln\|A_1 \cdots A_n\| \qquad (n \in \mathbb{N})$$

Theorem. If $L_n < 0$ for some n, then

- the following random sum exists with prob one:

$$x^* := \eta_0 + A_1\,\eta_1 + A_1 A_2\,\eta_2 + A_1 A_2 A_3\,\eta_3 + \cdots$$

- $\Psi^* :\stackrel{\mathscr{D}}{=} x^*$ is stationary and $(\mathcal{P}(\mathbb{R}^n), \Pi)$ is globally stable

Intuition for stationarity of $\Psi^* \stackrel{\mathscr{D}}{=} x^*$

We need to show that if $x_t \stackrel{\mathscr{D}}{=} x^*$, then $x_{t+1} \stackrel{\mathscr{D}}{=} x^*$

Equivalent: if (A, η) drawn independently, then

$$Ax^* + \eta \stackrel{\mathscr{D}}{=} \eta_0 + A_1 \eta_1 + A_1 A_2 \eta_2 + \cdots$$

True because

$$\begin{aligned} Ax^* + \eta &= \eta + A(\eta_0 + A_1 \eta_1 + A_1 A_2 \eta_2 + \cdots) \\ &= \eta + A\eta_0 + AA_1 \eta_1 + AA_1 A_2 \eta_2 + \cdots \\ &\stackrel{\mathscr{D}}{=} \eta_0 + A_1 \eta_1 + A_1 A_2 \eta_2 + A_1 A_2 A_3 \eta_3 + \cdots \end{aligned}$$

Example. Consider the vector autoregression model

$$x_{t+1} = Ax_t + \eta_{t+1} \quad \text{when } \eta_{t+1} := b + C\xi_{t+1}$$

The exponent L_n translates to

$$\frac{1}{n} \mathbb{E} \ln \|A_1 \cdots A_n\| = \frac{1}{n} \ln \|A^n\| = \ln \left\{ \|A^n\|^{\frac{1}{n}} \right\}$$

By Gelfand's formula,

$$\|A^n\|^{\frac{1}{n}} \to r(A) \quad (n \to \infty)$$

Hence $r(A) < 1$ implies $L_n < 0$ for large n

Example. Consider the GARCH(1, 1) volatility process

$$\sigma_{t+1}^2 = \alpha_0 + \sigma_t^2(\alpha_1 \xi_{t+1}^2 + \beta)$$

where

- $\{\xi_t\}$ is IID with $\mathbb{E}\xi_t^2 = 1$
- all parameters are positive

Ex. Show stability holds when $\mathbb{E} \ln(\alpha_1 \xi_{t+1}^2 + \beta) < 0$

A sufficient condition often used in the literature is $\alpha_1 + \beta < 1$

Ex. Show this condition is sufficient

Intermezzo: Heavy Tailed Distributions

Heavy tails matter for observed economic outcomes

- tail risk impacts asset prices
- Heavy tails in the wealth and income distributions shape our society / politics / welfare

Encountered frequently in social science

- city size distributions
- firm size distributions
- asset returns in high frequency data
- number of citations received by a given scientific paper

Power Laws

A random variable X is said to have a **power law** in the right tail if

$$\exists\, \alpha, c > 0 \text{ s.t. } \lim_{x \to \infty} x^{\alpha} \mathbb{P}\{X > x\} = c$$

Intuition:

- $\mathbb{P}\{X > x\}$ is proportional to $x^{-\alpha}$ for large x
- right tail decay is much slower than the Gaussian case

Example. The **Pareto** CDF takes the form

$$F(x) = \begin{cases} 1 - (\check{x}/x)^{\alpha} & \text{if } x \geqslant \check{x} \\ 0 & \text{if } x < \check{x} \end{cases}$$

With linear models, it's thin tails in $\implies$ thin tails out

That is, if

- $x_{t+1} = Ax_t + \eta_{t+1}$
- $\{\eta_t\} \overset{\text{IID}}{\sim} \varphi$ and φ has thin tails
- ψ^* is the stationary distribution of $\{x_t\}$

then ψ^* has thin tails

With random coefficient models, the story is different

Example. Consider the positive, scalar version

- $x_{t+1} = A_{t+1} x_t + \eta_{t+1}$
- $\{A_t\}$ and $\{\eta_t\}$ both positive and scalar

Theorem. (Kesten) If

- (some technical restrictions)
- There exists a positive constant α such that

$$\mathbb{E} A^\alpha = 1, \quad \mathbb{E} \eta^\alpha < \infty, \quad \text{and} \quad \mathbb{E}[A^\alpha \ln^+ A] < \infty$$

then there exists a random variable x^* on $\mathbb{R}_+$ such that

$$x^* \stackrel{\mathscr{D}}{=} A_t x^* + \eta_{t+1} \quad \text{and} \quad \lim_{x \to \infty} x^\alpha \mathbb{P}\{x^* > x\} = c$$

for some $c, \alpha > 0$

Sketch of proof (due to Gabaix): The influence of $\{\eta_t\}$ in $x_{t+1} = A_{t+1} x_t + \eta_{t+1}$ is insignificant when x_t is large

So

1. set $\eta_t \equiv 0$
2. let A_t have density g for all t
3. consider the stationary density ψ^* of $x_{t+1} = A_{t+1} x_t$

Fact. If A has density g on $(0, \infty)$ and $x > 0$, then the density of $Y = Ax$ is

$$f(y) = g\left(\frac{y}{x}\right)\frac{1}{x}$$

Hence density of $y := x_{t+1} = A_{t+1} x_t$ given $x_t = x$ is

$$\pi(x, y) = g\left(\frac{y}{x}\right)\frac{1}{x}$$

So

$$\psi^*(y) = \int \pi(x, y)\psi^*(x)\,\mathrm{d}x = \int g\left(\frac{y}{x}\right)\frac{1}{x}\psi^*(x)\,\mathrm{d}x$$

Ex. Show that $\psi^*(x) = kx^{-\alpha-1}$ is a solution for some constant k, provided that $\int g(t)t^\alpha\,\mathrm{d}t = 1$

...which is true by assumption (recall $\mathbb{E}A^\alpha = 1$)

A Nonlinear Scalar Model

Several models we study are scalar and have the form

$$X_{t+1} = \zeta_{t+1} g(X_t) + \eta_{t+1}$$

where

1. g is a Borel measurable function from $\mathbb{R}_+$ to itself,
2. $\{\zeta_t\}$ is IID on $\mathbb{R}_+$ with density ν
3. $\{\eta_t\}$ is IID on $\mathbb{R}_+$ with density φ
4. $\{\eta_t\}$ and $\{\zeta_t\}$ are independent

What is the density stochastic kernel?

What is the density of $Y := \zeta_{t+1} g(x) + \eta_{t+1}$?

Fact. If U has density φ_U and $Y = f(U)$ where f is continuously differentiable and strictly increasing, then the density of Y is

$$\varphi_Y(y) = \varphi_U(f^{-1}(y)) \left| \frac{\mathrm{d} f^{-1}(y)}{\mathrm{d} y} \right|$$

Hence density of $Y = \zeta g(x) + \eta$ given $\zeta = z$ is $y \mapsto \varphi(y - zg(x))$

(Here we take $\varphi(u) = 0$ whenever $u \leqslant 0$)

By the law of total probability

$$\pi(x, y) = \int \varphi(y - zg(x)) \nu(\mathrm{d} z)$$

Hence the marginal densities $\{\psi_t\}$ of

$$X_{t+1} = \zeta_{t+1} g(X_t) + \eta_{t+1}$$

obey

$$\psi_{t+1} = \psi_t \Pi$$

where

$$(\psi \Pi)(y) = \int \int \varphi(y - z g(x)) \nu(\mathrm{d}z) \psi(x) \, \mathrm{d}x$$

- A self-mapping on $\mathcal{D}$, the set of densities on $\mathbb{R}_+$
- When is $(\mathcal{D}, \Pi)$ is globally stable?

Regarding the process $X_{t+1} = \zeta_{t+1} g(X_t) + \eta_{t+1}$, we have:

Proposition. If

1. the density φ of η has finite first moment with $\varphi \gg 0$ and
2. there exist positive constants L and λ such that $\lambda < 1$ and

$$\mathbb{E} \zeta g(x) \leqslant \lambda x + L \qquad (x \geqslant 0)$$

then $(\mathcal{D}, \Pi)$ is globally stable, with unique stationary density ψ^*

If h is Borel measurable and $\int |h(x)| \psi^*(x) \, \mathrm{d}x < \infty$, then

$$\frac{1}{n} \sum_{t=1}^{n} h(X_t) \to \int h(x) \psi^*(x) \, \mathrm{d}x$$

with probability one as $n \to \infty$

The positive density restriction on φ is stronger than we need

Its role to generate **mixing**

- like irreducibility for finite state systems
- stops us getting "stuck" at "local attractors"

Example. Suppose

- $\zeta \equiv 1$ and $\eta \equiv 0$
- g has multiple fixed points

Then stability fails

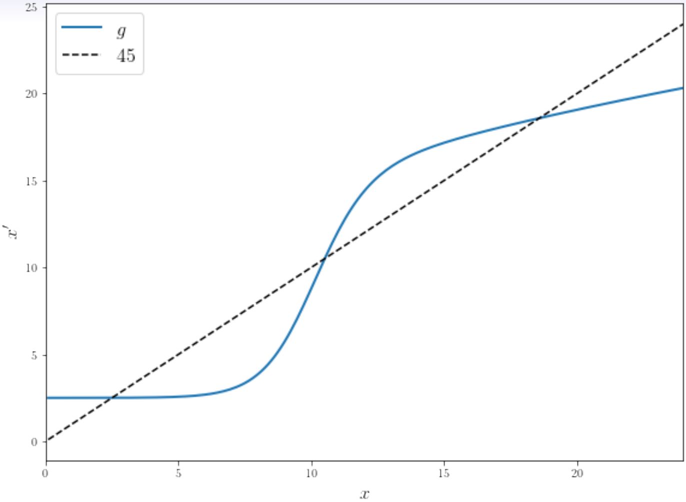

Figure: Dynamics with a degenerate shock and multiple fixed points

Even if

- ζ and η are permitted to have densities
- these densities have small supports

then local attractors will have permanent influence

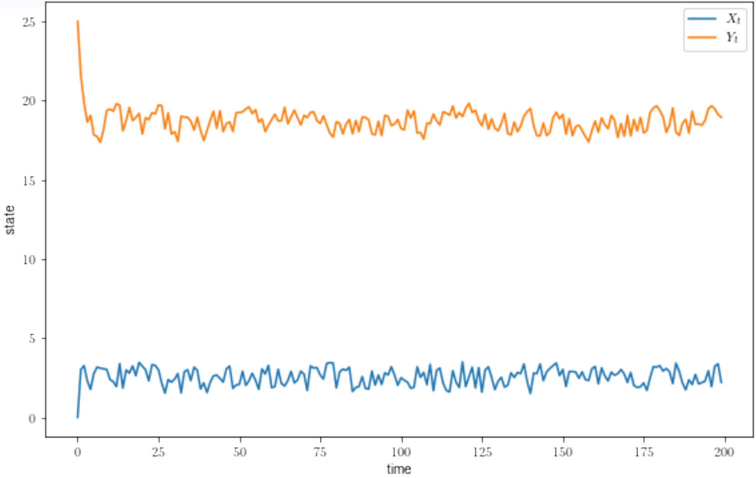

Figure: Time series with small shocks

Condition (ii) prevents $\{X_t\}$ from diverging to $+\infty$

When it holds we have

$$\begin{aligned}\mathbb{E}[X_{t+1} \mid X_t] &= \mathbb{E}[\zeta_{t+1} g(X_t) + \eta_{t+1} \mid X_t] \\ &= \mathbb{E}[\zeta] g(X_t) + \mathbb{E}[\eta] \leqslant \lambda X_t + K\end{aligned}$$

where $K := L + \mathbb{E}[\eta]$

Taking expectations of both sides gives

$$\mu_{t+1} \leqslant \lambda \mu_t + K$$

where μ_t is the mean of X_t for each t

Since $\lambda < 1$, the mean of X_t is bounded by $K/(1-\lambda)$

Wealth Dynamics

Let's examine a particular nonlinear stochastic model representing wealth dynamics

$$w_{t+1} = R_{t+1} s(w_t) + y_{t+1}$$

Simplifying assumptions

- All shocks independent across household
- Savings rule ad hoc (optimal rule to be treated soon)

How does the wealth distribution evolve?

Can our simple model replicate the data?

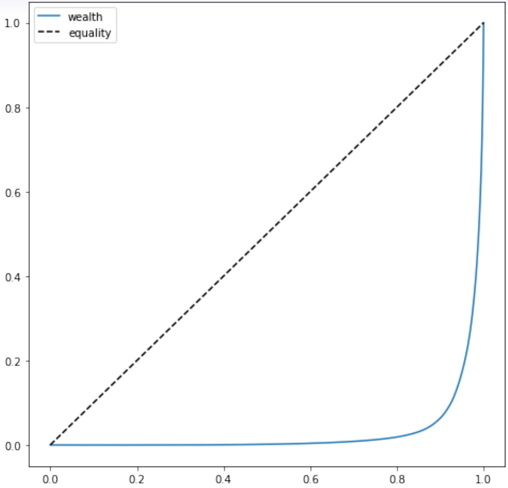

Figure: Lorenz curve, wealth distribution in the US (SCF 2016)

So let's look at our model of wealth dynamics

To repeat,

$$w_{t+1} = R_{t+1} s(w_t) + y_{t+1}$$

- y_{t+1} is IID with common density φ
- R_{t+1} is IID with common distribution ν

Baseline savings rule:

$$s(w) = \mathbb{1}\{w > \bar{w}\} s_0 w$$

where $\bar{w}$ and s_0 are positive parameters

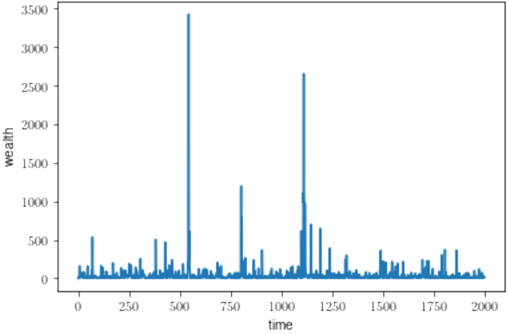

Figure: Time series for one household

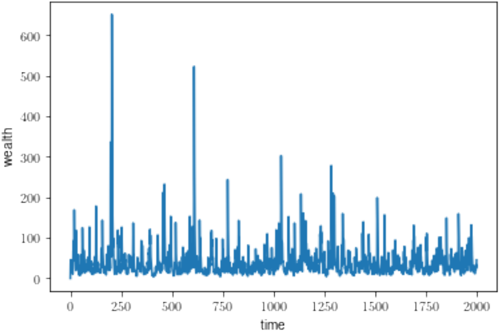

Figure: Holding $\{R_t\}$ at the constant $\mathbb{E}R_t$ means smaller spikes

Our wealth process is a version of the earlier model

$$X_{t+1} = \zeta_{t+1} g(X_t) + \eta_{t+1}$$

In particular, the stochastic density kernel is

$$\pi(w, w') := \int \varphi(w' - zs(w)) \nu(z) \, \mathrm{d}z$$

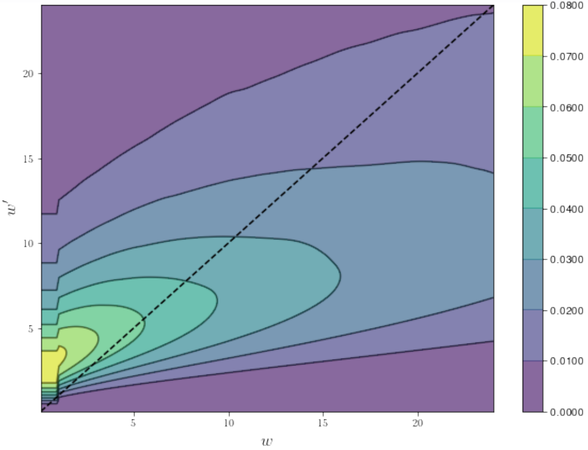

Figure: Stochastic 45 degree diagram for wealth dynamics

The wealth distribution process $\{\psi_t\}$ obeys

$$\psi_{t+1} = \psi_t \Pi$$

where

$$\begin{aligned}(\psi\Pi)(w') &= \int \pi(w, w')\psi(w)\,\mathrm{d}w \\ &= \int\int \varphi(w' - zs(w))\nu(\mathrm{d}z)\psi(w)\,\mathrm{d}w\end{aligned}$$

How does it evolve?

- analytical expressions for ψ_t not generally available
- but we can track it by simulation

Algorithm 1: Draws from the marginal distribution ψ_t

```
for i in 1 to m do
    draw w from the initial condition ψ_0 ;
    for j in 1 to t do
        draw R' and y' from their distributions ;
        set w = R' s(w) + y' ;
    end
    set w_t^i = w ;
end
return (w_t^1, ..., w_t^m)
```

Given $\{w_t^m\}$, we can to compute the empirical distribution

$$F_t^m(x) := \frac{1}{m} \sum_{i=1}^{m} \mathbb{1}\{w_t^i \leqslant x\}$$

An **unbiased** estimator of the CDF Ψ_t of w_t

$$\mathbb{E}[F_t^m(x)] = \frac{1}{m} \sum_{i=1}^{m} \mathbb{E}[\mathbb{1}\{w_t^i \leqslant x\}] = \frac{1}{m} m \mathbb{P}\{w_t \leqslant x\} = \Psi_t(x)$$

Also **consistent**, since the SLLN yields, with prob one,

$$\lim_{m \to \infty} F_t^m(x) = \mathbb{E}[\mathbb{1}\{w_t^i \leqslant x\}] = \mathbb{P}\{w_t \leqslant x\} = \Psi_t(x)$$

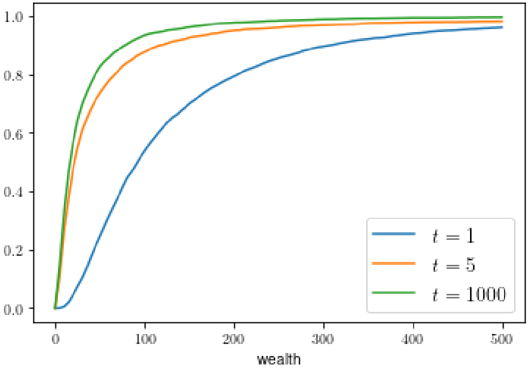

Figure: The empirical distribution F_m^t for different values of t