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Stochastic growth: asymptotic distributions*

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Summary. This note studies conditions under which sequences of state variables generated by discrete-time stochastic optimal accumulation models have law of large numbers and central limit properties. Productivity shocks with unbounded support are considered. Instead of restrictions on the support of the shock, an "average contraction" property is required on technology.

Keywords and Phrases: Stochastic growth, Law of large numbers, Central limit theorem.

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1 Introduction

Dynamic properties of the one-sector stochastic optimal growth model with concave production technology were first studied in the well-known paper of Brock and Mirman [4]. Using standard Inada-type assumptions, they showed that there exists a unique and globally stable equilibrium whenever the shock that perturbs production is supported on a closed bounded interval of the positive real numbers. Subsequently, the asymptotic behavior of the concave one-sector model with bounded shock was studied by many authors [12, 13, 5, 14, 6, 10, 17, 8, 1]. More general proofs were established, and various complications were incorporated.

Stochastic models that have an equilibrium or steady state distribution may also have asymptotic statistical properties related to the existence of a steady state distribution, such as convergence of sample averages from time series to the mean

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of this limiting distribution, or asymptotic normality of the partial sums. The first question concerns a law of large numbers (LLN) result, while the latter pertains to the central limit theorem (CLT).

The importance of these questions can be summarized as follows. If an LLN condition holds, then it is possible to test a given theoretical model by comparing the mean of the limiting distribution with a sample average from a sufficiently large data set generated by the system under study. Conversely, suppose that an expression is available for the mean of the hypothetical model in terms of its parameters. Then the implied equality of this expression and the sample mean calculated from data provides a consistent method for estimating parameter values. If, in addition, a CLT result is available, then inference can be drawn as to the likelihood of values in the parameter space.

It is known that both the LLN and CLT results are realized for the general discrete-time concave stochastic optimal growth model when the shock has compact support [17,7,2,3]. Central to the proofs is boundedness of the productivity shock, which, in combination with Inada conditions, allows the state space to be taken to be compact.

Recently, general conditions for existence, uniqueness and stability of equilibrium in the Brock-Mirman model without restrictions on the support of the productivity shock were found [15]. When the influence of the shock is not bounded, however, compactness of the state space typically fails. In this case it is not clear whether LLN and CLT properties are available.

Here we begin to address this question by introducing a methodology and studying some specific parameterizations. The techniques are based on results recently obtained by Łoskot and Rudnicki [9], who study the LLN and CLT properties of pertubed dynamical systems on Polish space.

Section 2 formulates the problem and gives the major definitions. Section 3 provides a general stability result. Section 4 gives applications. Section 5 gives proofs.

2 The model

Let (X, ρ) be a metric space, and let $\mathcal{B}$ be the Borel subsets of X. Real $\mathcal{B}$-measurable functions on X are referred to as Borel functions. A finite Borel measure on X is a nonnegative, countably additive set function $\mu\colon \mathcal{B} \to \mathbb{R}$. The vector lattice of finite Borel measures is denoted by $\mathcal{M}$. All integrals are taken over the space X unless otherwise stated.

We consider a growth model evolving on state space X. Preferences, technology, market conditions and other primitives of the model imply a transition rule that associates a current state value x_t and a state of nature-a random variable ϵ_t-with next-period state x_{t+1}.

Formally, let $(S, \mathcal{S}, \mathbf{P})$ be a probability space, and let $\epsilon\colon S \to X$ be an $(\mathcal{S}, \mathcal{B})$-measurable random variable. At time t, a point $s_t \in S$ is drawn independently according to $\mathbf{P}$, and mapped into X by ϵ. As anticipated by the above notation, the realized value $\epsilon(s_t)$ is written simply as ϵ_t.

Let the transition rule be denoted by T. That is,

$$T\colon X \times X \ni (x_t, \epsilon_t) \mapsto x_{t+1} \in X. \tag{1}$$

Corresponding to the random variable ϵ is a finite Borel measure $\psi \in \mathcal{M}$ defined at $B \in \mathcal{B}$ by $\psi(B) = \mathbf{P}[\epsilon^{-1}(B)]$. The measure ψ is called the *distribution* of ϵ, and $\int_S f[\epsilon(s)]\mathbf{P}(ds) = \int_X f(z)\psi(dz)$ for any Borel function f.

Analogous to the definition in Brock and Mirman [4, p. 492], an *equilibrium* for the economy (1) is a probability measure $\phi \in \mathcal{M}$ that satisfies

$$\int \left[\int \mathbf{1}_B[T(x,z)]\psi(dz) \right] \phi(dx) = \phi(B) \tag{2}$$

for all $B \in \mathcal{B}$.[1] The equilibrium is unique if there exists no other point in $\mathcal{M}$ satisfying (2).

Suppose that the growth model (1) has a unique equilibrium $\phi \in \mathcal{M}$. For the purposes of this paper, (1) is said to satisfy the law of large numbers if, for any Lipshitz function $g\colon X \to \mathbb{R}$,

$$\frac{1}{N} \sum_{t=0}^{N-1} g(x_t) \to \int g(x)\phi(dx) \tag{3}$$

$\mathbf{P}$-almost surely as $N \to \infty$.[2]

The economy is said to have the central limit property if, for any g as above,

$$\frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} g(x_t) \to N(m, \sigma^2) \tag{4}$$

in distribution, where $N(m, \sigma^2)$ is a normal distribution with mean $m = \int g(x)\phi(dx)$ and variance $\sigma^2 \geq 0$.

3 Results

Łoskot and Rudnicki [9] consider stochastic models that satisfy the following contraction condition.

Definition 1. *The pair (T, ψ), where T is the map in (1) and ψ is the distribution of the shock ϵ, is called an* average contraction *on (X, ρ) if there exists a Borel function $\lambda\colon X \to \mathbb{R}$ such that $E(\lambda) = \int \lambda(z)\psi(dz) < 1$ and*

$$\rho(T(x,z), T(x',z)) \leq \lambda(z)\rho(x,x'), \quad \forall x, x', z \in X.$$

By adapting the results of Łoskot and Rudnicki and using additional restrictions on the space X, we obtain the following stability condition.

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[1] Here $\mathbf{1}_B\colon X \to \{0,1\}$ is the characteristic function of B.

[2] A real function g on X is called *Lipshitz* if there exists a constant λ such that $|g(x) - g(x')| \leq \lambda\rho(x,x')$ for any $x, x' \in X$.

Theorem 1. *Let T be the map in (1). Let the state space (X, ρ) be both locally compact and σ-compact.*[3] *If the growth model defined by the law of motion T and the distribution of the shock ψ is an average contraction, and if there exists at least one point $\bar{x} \in X$ such that*

$$\int \rho(\bar{x}, T(\bar{x}, z))\psi(dz) < \infty, \tag{5}$$

then there exists a unique stochastic equilibrium $\phi \in \mathcal{M}$ satisfying (2), and, in addition, the model satisfies the law of large numbers property (3). If, moreover, $E(\lambda^2) < 1$ and

$$\int [\rho(\bar{x}, T(\bar{x}, z))]^2 \psi(dz) < \infty \tag{6}$$

holds, then the model also satisfies the central limit property (4).

The proof (Section 5) is a straightforward consequence of the results of Łoskot and Rudnicki. The only technical difficulty is to verify that the steady state notion used by Łoskot and Rudnicki is equivalent to the definition (2), which is standard in the economic literature. This can be done under local and σ-compactness of the state space, as was assumed in the theorem.

4 Applications

Let $X = (0, \infty)$. Consider the one-sector optimal growth problem

$$\max \mathrm{E}\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right] \tag{7}$$

$$\text{s.t. } k_{t+1} = f(k_t, \epsilon_t) - c_t \tag{8}$$

where $c_t \in X$ is consumption, $k_t \in X$ is capital per head, $\beta \in (0, 1)$ is a discount factor, and $f\colon X \times X \to X$ and $u\colon X \to \mathbb{R}$ are the production and utility functions respectively [4, pp. 484–488]. The utility function u is assumed to satisfy $u'(0) = \infty$, which assures interiority of solutions, and therefore eliminates the possibility of zero savings or consumption. The shocks $\epsilon_t \in X$ are independent draws from $\mathbf{P}$ as before.

The solution to the planning problem, if it exists, is an optimal policy $\pi\colon X \to X$, which associates realized output $f(k_t, \epsilon_t)$ with optimal current consumption c_t. Optimal consumption $\pi(f(k_t, \epsilon_t))$ can then be substituted into (8) to obtain the closed loop law of motion for the system, which is in the form of (1). A unique and well-defined stochastic process is specified by this law and any initial condition $k_0 \in X$. The process so generated is called an *optimal program*.

[3] Recall that a topological space is called *locally compact* if every point in the space has a neighborhood with compact closure, and *σ-compact* if every open set can be obtained as the union of a countable number of compact sets.

Example 1. Consider first the unit-elastic decreasing returns model $u(c) = \ln c$, $f\colon (k, \epsilon) \mapsto Ak^\alpha \epsilon$, $A > 0$, $\alpha \in (0, 1)$. For such a specification, the optimal policy consumes a fraction $1 - \alpha\beta$ of realized output $Ak^\alpha \epsilon$, implying the law of motion

$$k_{t+1} = \alpha\beta A k_t^\alpha \epsilon_t. \tag{9}$$

Define a binary ρ on $X \times X$ by $\rho(x, y) = |\ln x - \ln y|$. Evidently ρ is a metric on X. Moreover, the space (X, ρ) is isometrically isomorphic to $\mathbb{R}$ under the mapping $x \mapsto \ln x$ when the latter space is endowed with its usual Euclidean metric. Hence (X, ρ) is both locally and σ-compact.

It can be verified that (9) is an average contraction on (X, ρ) for every random variable ϵ. If $E|\ln \epsilon|$ is finite, then condition (5) of Theorem 1 holds for $\bar{x} = 1$, implying the existence of a unique stochastic equilibrium $\phi \in \mathcal{M}$, and the LLN result (3) for the process $(k_t)_{t\geq 0}$.

Evidently $E(\lambda^2) < 1$ also holds. If, in addition, $E[(\ln \epsilon)^2]$ is finite, then (6) is satisfied for $\bar{x} = 1$, and the CLT result (4) obtains.

Remark 1. The conditions $E|\ln \epsilon| < \infty$ and $E[(\ln \epsilon)^2] < \infty$ can be viewed as restrictions on the left- and right-hand tails of the distribution. See the discussion in Stachurski [16, Remark 4.1].

Example 2. The second example is from Mirman and Zilcha [13, Example A, p. 333]. The state space (X, ρ), the shock ϵ, the discount factor β, the productivity parameter A and utility $u(c) = \ln c$ are as before. Let α be a Borel function from X into $(0, 1)$. The production function is $(k, \epsilon) \mapsto Ak^{\alpha(\epsilon)}$. For such a specification, the law of motion is

$$k_{t+1} = \bar{\alpha}\beta A k_t^{\alpha(\epsilon_t)}, \quad \bar{\alpha} = \int_S \alpha[\epsilon(s)]\mathbf{P}(ds). \tag{10}$$

Once again, the system is an average contraction on X under ρ, this time using $\lambda(z) = \alpha(z)$. Note that in this case (5) holds for *any* shock ϵ when $\bar{x} = 1$. Hence for any strictly positive initial condition k_0, a unique equilibrium distribution ϕ exists and the LLN condition (3) holds.

Moreover, $E(\lambda^2) < 1$, and (6) holds for any shock ϵ when $\bar{x} = 1$, implying that the CLT condition (4) also holds.

5 Proofs

This section contains the proof of Theorem 1. Throughout, C_b denotes the Banach lattice of continuous bounded real functions on X, and $C_0 \subset C_b$ denotes the continuous real functions with compact support. In what follows, the scalar product notation is used for integration. Thus, for Borel function $f\colon X \to \mathbb{R}$ and $\mu \in \mathcal{M}$, we write $\langle f, \mu \rangle$ for $\int f d\mu$.

For $f \in C_0$ and fixed $\mu \in \mathcal{M}$,

$$\Lambda(f) = \langle f, \mu \rangle \tag{11}$$

defines a positive linear functional Λ on C_0. Denote by C_0^* the class of positive linear functionals on C_0. We will make use of the well-known fact that the association $\mu \mapsto \Lambda$ from the finite Borel measures $\mathcal{M}$ to the linear functionals C_0^* defined by (11) is one-to-one [, Theorems 38.3 and 38.4].

Łoskot and Rudnicki [9, Theorems 1 and 3] proved that when (T, ψ) is an average contraction, (X, ρ) is complete and separable, and (5)–(6) holds, then there exists a unique distribution $\phi \in \mathcal{M}$ such that

$$\int \left[\int f[T(x,z)]\psi(dz) \right] \phi(dx) = \int f(x)\phi(dx), \quad \forall f \in C_b, \tag{12}$$

and, moreover, the LLN and CLT results (3) and (4) both hold for ϕ.

All of the Łoskot-Rudnicki conditions are satisfied under the assumptions of Theorem 1. It remains only to verify that ϕ in (12) is also a Brock-Mirman equilibrium in the sense of (2).

Lemma 1. *Let (X, ρ) be a locally and σ-compact metric space, and let ϕ be any finite Borel measure on X. If ϕ satisfies (12), then it also satisfies (2).*

Proof. Define an operator P from the set of all bounded Borel functions $f: X \to \mathbb{R}$ into itself by

$$(Pf)(x) = \int f[T(x,z)]\psi(dz), \quad x \in X.$$

Define in addition an operator P^* from the space of finite measures $\mathcal{M}$ into itself by

$$(P^*\mu)(B) = \int \int \mathbf{1}_B[T(x,z)]\psi(dz)\mu(dx), \quad B \in \mathcal{B}.$$

The operator P^* is adjoint to P in the sense that

$$\langle f, P^*\mu \rangle = \langle Pf, \mu \rangle \tag{13}$$

for every bounded Borel function f and every finite Borel measure μ. To see this, pick any $\mu \in \mathcal{M}$. Evidently (13) holds when $f = \mathbf{1}_B$. By linearity of the inner product, (13) also holds when f is a step function taking only finitely many values. This can be extended from step functions to any bounded nonnegative Borel function by pointwise approximation and a monotone convergence result in the usual way. Linearity then implies the result for an arbitrary bounded Borel function f, which can always be written as the difference between two nonnegative parts.

Assume now that ϕ satisfies (12). Then

$$\langle Pf, \phi \rangle = \langle f, \phi \rangle, \quad \forall f \in C_b.$$

Therefore,

$$\langle Pf, \phi \rangle = \langle f, \phi \rangle, \quad \forall f \in C_0. \tag{14}$$

Since ϕ is a finite Borel measure, and since each $f \in C_0$ is a bounded Borel function, together (13) and (14) imply that

$$\langle f, P^*\phi \rangle = \langle f, \phi \rangle, \quad \forall f \in C_0. \tag{15}$$

This says precisely that the positive linear functionals on C_0 generated by the two measures $P^*\phi$ and ϕ in the manner of (11) are identical. Given that $P^*\phi$ and ϕ are finite Borel measures, and that the association (11) from $\mathcal{M}$ to C_0^* is one-to-one, this implies that the representing measures $P^*\phi$ and ϕ are identical. This is equivalent to stating that ϕ satisfies (2), and completes the proof of the lemma.

References

1. Aliprantis, C.D., Burkinshaw, O.: Principles of real analysis. New York: Academic Press 1998
2. Amir, R.: A new look at optimal growth under uncertainty. Journal of Economic Dynamics and Control **22**, 67–86 (1997)
3. Amir, R., Evstigneev, I.V.: A functional central limit theorem for (2000)
4. Bhattacharya, R., Majumdar, M.: On a class of stable random dynamical systems: Theory and applications. Journal of Economic Theory **96**, 208–229 (2001)
5. Brock, W.A., Mirman, L.: Optimal economic growth and uncertainty: The discounted case. Journal of Economic Theory **4**, 479–513 (1972)
6. Brock, W.A., Majumdar, M.: Global asymptotic stability results for multisector models of optimal growth under uncertainty when future utilities are discounted. Journal of Economic Theory **18**, 225–243 (1978)
7. Donaldson, J.B., Mehra, R.: Stochastic growth with correlated production shocks. Journal of Economic Theory **29**, 282–312 (1983)
8. Evstigneev, I. V., Flåm, S. D.: The turnpike property and the central limit theorem in stochastic models of economic dynamics. In: Statistics and control of stochastic processes, pp. 63–101, Kabanov, Yu M., Rozovskii, B. L., Shiryaev, A. N. (Eds.), London: World Scientific 1997
9. Hopenhayn H.A., Prescott, E.C.: Stochastic monotonicity and stationary distributions for dynamic economies. Econometrica **60**, 1387–1406 (1992)
10. Łoskot, K., Rudnicki, R.: Limit theorems for stochastically perturbed dynamical systems. Journal of Applied Probability **32**, 459–469 (1995)
11. Majumdar, M., Zilcha, I.: Optimal growth in a stochastic environment: some sensitivity and turnpike results. Journal of Economic Theory **43**, 116–133 (1987)
12. Mirman, L.J.: On the existence of steady state measures for one sector growth models with uncertain technology. International Economic Review **13**, 271–286 (1972)
13. Mirman, L.J.: The steady state behavior of a class of one sector growth models with uncertain technology. Journal of Economic Theory **6**, 219–242 (1973)
14. Mirman, L.J., Zilcha, I.: On optimal growth under uncertainty. Journal of Economic Theory **11**, 329–339 (1975)
15. Razin, A., Yahav, J.A.: On stochastic models of economic growth. International Economic Review **20**, 599–604 (1979)
16. Stachurski, J.: Stochastic optimal growth with unbounded shock. Journal of Economic Theory, (forthcoming) (2002)
17. Stachurski, J.: Log-linearization of perturbed dynamical systems, with applications to optimal growth. University of Melbourne, Department of Economics Research Paper No. 788, 2001
18. Stokey, N.L., Lucas, R.E., Prescott, E.C.: Recursive methods in economic dynamics. Cambridge, MA: Harvard University Press 1989