Systemic Risk in Financial Systems: Properties of Equilibria

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ABSTRACT. Eisenbern and Noe (2001) analyze systemic risk for financial institutions linked by a network of liabilities. They show that the solution to their model is unique when the financial system is satisfies a regularity condition involving risk orbits. We show that this condition is not needed: a unique solution always exists.

Keywords: Systemic risk, uniqueness

1. INTRODUCTION

Eisenberg and Noe (2001) analyzed systemic risk for institutions linked by a complex network of liabilities. The fundamental importance of their work was brought to the fore by the financial crisis of 2007–2008, where the solvency of individual firms became unclear due to interlocking financial obligations. In such settings, defaults can trigger negative feedback loops, where restructuring compromises the balance sheets of other firms in the network, restricting their ability to make payments to debt holders, and forces more rounds of restructuring.

Because of these cyclical interactions in the networks, obtaining a system of payments that clears the market is a fixed point problem. While existence of a fixed point is easy to show, uniqueness is challenging. Eisenberg and Noe (2001) show that uniqueness holds when the financial system is "regular." This condition has become common in the literature (see, e.g., Cifuentes et al. (2005) or Feinstein et al. (2018)). We prove

it is not needed: the limited liability and priority conditions in Eisenberg and Noe (2001) are themselves enough to uniquely identify an equilibrium clearing vector.[1]

2. Results

If $a = (a(i))_{i=1}^n$ and $b = (b(i))_{i=1}^n$ are vectors in $\mathbb{R}^n$, then $a \leqslant b$ means $a(i) \leqslant b(i)$ for all i. We write $a \ll b$ if $a(i) < b(i)$ for all i. The symbol $[a, b]$ represents the order interval $\{x \in \mathbb{R}^n : a \leqslant x \leqslant b\}$. A Markov matrix is a nonnegative square matrix with unit row sums. For a given Markov matrix Π, we call j *accessible* from i under Π if either $j = i$ or $\Pi^k(i, j) > 0$ for some $k \in \mathbb{N}$.

As in Eisenberg and Noe (2001), a *financial system* is a set of nodes $I := \{1, 2, \ldots, n\}$, an $n \times n$ matrix $\Pi = (\Pi(i, j))$ of relative liabilities, a vector $\bar{p} = (\bar{p}(i)) \in \mathbb{R}_+^n$ of nominal obligations ($\bar{p}(i)$ is the sum of all nominal obligations held by node i) and a vector $e = (e(i)) \in \mathbb{R}_+^n$ of external cash flows. The matrix of relative liabilities is a Markov matrix. Following Eisenberg and Noe (2001), we assume that $\bar{p} \gg 0$, so that, for any given node, the total sum of liabilities to other nodes is not zero.[2]

A *clearing vector* for a financial system $(I, \Pi, \bar{p}, e)$ is a vector of payments $p \in [0, \bar{p}]$ satisfying the limited liability restriction

$$p(j) \leqslant \sum_{i \in I} p(i)\Pi(i, j) + e(j) \tag{1}$$

and the absolute priority condition

$$p(j) = \sum_{i \in I} p(i)\Pi(i, j) + e(j) \quad \text{or} \quad p(j) = \bar{p}(j) \tag{2}$$

for all $j \in I$. Combining these two restrictions and writing them in vector form (cf. Eisenberg and Noe (2001), p. 240), the set of clearing payment vectors is seen to coincide with the set of fixed points of the mapping $p \mapsto \Phi p$ on $[0, \bar{p}]$ defined by

$$\Phi p := (p\Pi + e) \wedge \bar{p}. \tag{3}$$

In (3) and below, all n-vectors are treated as row vectors.

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[1]The only exception is an extreme case where no firm in the entire network has any cash at all. This case is discussed in Remark 2.1 below.

[2]This is required for Π to have unit row sums. See p. 239 of Eisenberg and Noe (2001).

Remark 2.1. Limited liability and absolute priority cannot by themselves pin down outcomes for the extreme case where every firm in the network has zero operating cash.[3] In what follows, we adopt the convention $p = 0$ when $e = 0$. This is natural if firms cannot raise new loans to meet their liabilities when $e = 0$, since a payment sequence cannot be initiated without outside capital.

In Eisenberg and Noe (2001), the *risk orbit* of a given node i is the set of all nodes that are accessible from i. The financial system is called *regular* if the risk orbit of every node contains at least one j with $e(j) > 0$. A system where regularity fails is shown below. Suppose $e(1) = 1$ and $e(2) = e(3) = 0$. Arrows represent nonzero liabilities. The risk orbit from node 2 is $\{2, 3\}$. Regularity fails because $e(2) = e(3) = 0$.
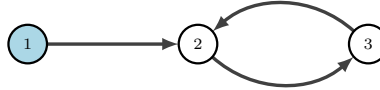
Fortunately, regularity is irrelevant for uniqueness, as the next theorem shows.

Theorem 2.1. *Every financial system has exactly one clearing payment vector.*

To prove Theorem 2.1, we begin with a lemma. In the lemma, we say that node j in a financial system $(I, \Pi, \bar{p}, e)$, is *cash accessible* if there exists an $i \in I$ such that $e(i) > 0$ and j is accessible from i.

Lemma 2.2. *If every node in $S = (I, \Pi, \bar{p}, e)$ is cash accessible, then S has a unique clearing vector p^*. Moreover, $p^* \gg 0$ and $\Phi^k p \to p^*$ as $k \to \infty$ when $0 \leqslant p \leqslant \bar{p}$.*

Proof. Let S be as described. By the fixed point theorem in the appendix (Theorem A.1), it suffices to show that Φ is an increasing concave self-map on $[0, \bar{p}]$ with $\Phi^k 0 \gg 0$ for some $k \in \mathbb{N}$. As confirmed in Eisenberg and Noe (2001), Φ is increasing and concave, so only the last statement needs to be verified. To this end, we set

$$\delta := \frac{1}{n^2} \cdot \min\left\{\{\bar{p}(i) : i \in I\} \cup \{e(i) : i \in I \text{ s.t. } e(i) > 0\}\right\}.$$

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[3]For example, if ψ is a stationary distribution for Π, λ is a constant in $[0,1]$, $p = \lambda\psi$ and $\bar{p} = \psi$, then $e = 0$ implies $\Phi p = (\lambda\psi\Pi + e) \wedge \bar{p} = (\lambda\psi) \wedge \psi = \lambda\psi = p$. Since λ was arbitrary in $[0,1]$, there is a continuum of equilibria.

Let $\hat{e}$ be defined by $\hat{e}(i) = 1$ if $e(i) > 0$ and zero otherwise. We claim that, for all $m \leqslant n$,

$$\Phi^m 0 \geqslant \delta(\hat{e} + \hat{e}\Pi + \cdots + \hat{e}\Pi^{m-1}). \tag{4}$$

This holds at $m = 1$ because $\Phi 0 = e \wedge \bar{p} \geqslant \delta\hat{e}$. Now suppose (4) holds at some $m \leqslant n - 1$. Then, since Φ is increasing, we obtain

$$\Phi^{m+1} 0 \geqslant (\delta(\hat{e} + \hat{e}\Pi + \cdots + \hat{e}\Pi^{m-1})\Pi + e) \wedge \bar{p}$$
$$\geqslant (\delta(\hat{e} + \hat{e}\Pi + \cdots + \hat{e}\Pi^m)) \wedge \bar{p}$$

Since $\hat{e} + \hat{e}\Pi + \cdots + \hat{e}\Pi^m \leqslant n^2 \mathbb{1}$, where $\mathbb{1}$ is a vector of ones, and since $(\delta n^2 \mathbb{1}) \leqslant \bar{p}$ by the definition of δ, we have $\Phi^{m+1} 0 \geqslant \delta(\hat{e} + \hat{e}\Pi + \cdots + \hat{e}\Pi^m)$. This argument confirms that (4) holds for all $m \leqslant n$.

We now claim that $\Phi^n 0 \gg 0$. In view of (4), it suffices to show that, for any $j \in I$, there exists a $k < n$ with $(\hat{e}\Pi^k)(j) = \sum_{i \in I} \hat{e}(i)\Pi^k(i,j) > 0$. Since every node in S is cash accessible, we know there exists an $i \in I$ with $e(i) > 0$ and j is accessible from i. For this i we can choose $k \in \mathbb{N}$ with $k < n$ and $\Pi^k(i,j) = \hat{e}(i)\Pi^k(i,j) > 0$. We conclude that $\Phi^n 0 \gg 0$, as claimed. $\qquad\square$

In what follows, a subset $\mathbb{J}$ of I is called *absorbing* if no element of its complement $\mathbb{J}^c := \{i \in I : i \notin \mathbb{J}\}$ is accessible from $\mathbb{J}$. Also, for a given vector v on I and some $\mathbb{J} \subset I$, we write $v \,|\, \mathbb{J}$ for the restriction of v to $\mathbb{J}$. For matrix M on $I \times I$ we write $M \,|\, \mathbb{J}$ for the restriction of M to $\mathbb{J} \times \mathbb{J}$.

Proof of Theorem 2.1. As pointed out by Eisenberg and Noe (2001), the operator Φ is increasing and concave on $[0, \bar{p}]$. It follows from the increasing property and Tarski's fixed point theorem that at least one clearing vector always exists.[4] The remainder of the proof focuses on uniqueness.

Let P be the set of all nodes in I that are cash accessible. Let A be all i in P^c such that P is accessible from i. Let N be all i in P^c such that P is not accessible from i. Note that $I = P \cup A \cup N$ and that these sets are disjoint.

The set N is an absorbing set, since, by definition, P is not accessible from N, and A cannot be accessible because otherwise P would also be accessible. The set P is also

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[4]In fact Φ is a continuous self-map on the convex compact set $[0, \bar{p}]$, so Brouwer's fixed point theorem gives the same conclusion.

absorbing because, if $j \in P^c$ is accessible from some $i \in P$, then j is cash accessible. But then $j \in P$, which is a contradiction.

Iterating k times on (1) gives

$$p \leqslant e + e\Pi + e\Pi^2 + \cdots + e\Pi^{k-1} + p\Pi^k. \tag{5}$$

If j is in A, then j is not cash accessible, so $e(j) = 0$ and $(e\Pi^m)(j) = 0$ for all m. Hence (5) reduces to

$$p(j) \leqslant (p\Pi^k)(j) \leqslant (\bar{p}\Pi^k)(j) = \sum_{i \in I} \bar{p}(i)\Pi^k(i,j) = \sum_{i \in A} \bar{p}(i)\Pi^k(i,j). \tag{6}$$

The last equality uses the fact that both P and N are absorbing sets. Since P is accessible from every element of A, when $i, j \in A$ we have $\Pi^k(i,j) \to 0$ as $k \to \infty$.[5] By taking k large in (6), we see that $p(j) = 0$ for all $j \in A$.

Since N is absorbing, $\Pi \,|\, N$ is a Markov matrix and $(N, \Pi \,|\, N, \bar{p} \,|\, N, e \,|\, N)$ is itself an independent financial system.[6] Moreover, P contains all cash accessible nodes so no element of N is cash accessible and, in particular, $e \,|\, N = 0$. Hence $p \,|\, N = 0$ by the convention in Remark 2.1.

It remains only to treat nodes in P. Since P is absorbing, $\Pi \,|\, P$ is a Markov matrix and $(P, \Pi \,|\, P, \bar{p} \,|\, P, e \,|\, P)$ is also an independent financial system.[7] Moreover, every node in P is cash accessible, so, by Lemma 2.2, a unique clearing vector $p^* \gg 0$ exists on P. After extending to all nodes by setting $p^*(i) = 0$ for all $i \notin P$, we have a unique clearing vector. $\qquad\square$

Appendix A. Remaining Proofs

In what follows, a self-map F from a subset Y of $\mathbb{R}^d$ to itself is called *globally stable* if F has a unique fixed point $\bar{y}$ in Y and $F^m y \to \bar{y}$ as $m \to \infty$ for all $y \in Y$. We use the following fixed point theorem, which slightly modifies Theorem 3.1 of Du (1990). (See also Corollary 2.1.1. of Zhang (2012).)

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[5] A Markov chain started at $i \in A$ leaves for P with $\varepsilon > 0$ probability every $n = |I|$ periods. Since P is absorbing it never returns. Hence the probability that the chain hits $j \in A$ after k periods converges to zero with k.

[6] Although nodes in N may have inbound links from A, we have just shown that payments from A are zero. At the same time, there are no inbound links from P, since P is aborbing.

[7] Again, while nodes in P may have inbound links from A, we have just shown that payments from A are zero.

Theorem A.1. *Let A be an increasing self-map on $[0, b] \subset \mathbb{R}^d$. If A is concave and there exists an $n \in \mathbb{N}$ such that $A^n 0 \gg 0$, then A is globally stable on $[0, b]$.*

Proof. Theorem 3.1 of Du (1990) implies that any increasing concave self-map F on $[0, b]$ satisfying $F0 \gg 0$ is globally stable. Since compositions of increasing concave operators are increasing and concave, this implies that A^n is globally stable on $[0, b]$. Denote its fixed point by $\bar{v}$. Since $\{A^m 0\}_{m \in \mathbb{N}}$ is monotone increasing and since the subsequence $\{A^{mn} 0\}_{m \in \mathbb{N}}$ converges up to $\bar{v}$ as $m \to \infty$, we must have $A^m 0 \to \bar{v}$. A similar argument gives $A^m b \to \bar{v}$. For any $v \in [0, b]$ we have $A^m 0 \leqslant A^m v \leqslant A^m b$, so $A^m v \to \bar{v}$ as $m \to \infty$.

The last step is to show that $\bar{v}$ is the unique fixed point of A. From Tarski's fixed point theorem we know that at least one fixed point exists. Now suppose $v \in [0, b]$ is such a point. Then $v = A^m v$ for all m. At the same time, $A^m v \to \bar{v}$ by the results just established. Hence $v = \bar{v}$. The proof is now complete. $\qquad \square$

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