ECON2125/4021/8013

Lecture 9

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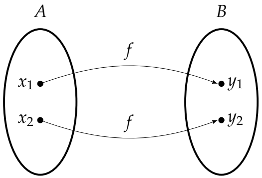
Announcements

- Preliminary midterm exam date: April 23rd
- Solved exercises up on GitHub
- Extended office hours for tutors
  - 4:00-6:00pm on Friday for Guanlong
  - 3:00-5:00pm on Friday for Qingyin
- Proofs / logic / sets reference, if you want one
  - Simon and Blume, appendix A1
  - Sydsaeter and Hammond, Chapter 1
- Linear algebra reference, if you want one
  - "Linear Algebra" by David Lay (expensive but good)

Reminder I

Suppose we want to find the x that solves $f(x) = y$

The ideal case is when f is a bijection



Equivalent:

- f is a bijection
- each $y \in B$ has a unique preimage
- $f(x) = y$ has a unique solution x for each y

Reminder II

Let T be a linear function from $\mathbb{R}^N$ to $\mathbb{R}^N$

We saw that in this case all of the following are equivalent:

1. T is a bijection
2. T is onto
3. T is one-to-one
4. $\ker(T) = \{\mathbf{0}\}$
5. $V := \{T\mathbf{e}_1, \ldots, T\mathbf{e}_N\}$ is linearly independent

We then say that T is nonsingular ($=$ linear bijection)

Linear Equations

Let's look at solving linear equations such as $\mathbf{Ax} = \mathbf{b}$

We start with the "best" case:

$$\text{number of equations} = \text{number of unknowns}$$

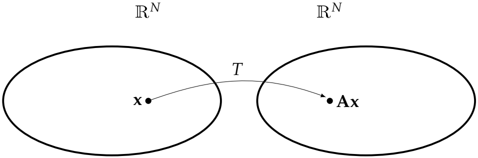
Thus,

- Take $N \times N$ matrix $\mathbf{A}$ and $N \times 1$ vector $\mathbf{b}$ as given
- Search for an $N \times 1$ solution $\mathbf{x}$

But does such a solution exist? If so is it unique?

The best way to think about this is to consider the corresponding linear map

$$T\colon \mathbb{R}^N \to \mathbb{R}^N, \qquad T\mathbf{x} = \mathbf{A}\mathbf{x}$$



Equivalent:

1. $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any given $\mathbf{b}$
2. $T\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for any given $\mathbf{b}$
3. T is a bijection

We already have conditions for linear maps to be bijections

Just need to translate these into the matrix setting

Recall that T called nonsingular if T is a linear bijection

We say that $\mathbf{A}$ is **nonsingular** if T is nonsingular

Equivalent:

- $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is a bijection from $\mathbb{R}^N$ to $\mathbb{R}^N$

We now list equivalent conditions for nonsingularity

Let $\mathbf{A}$ be an $N \times N$ matrix

Fact. All of the following conditions are equivalent

1. $\mathbf{A}$ is nonsingular
2. The columns of $\mathbf{A}$ are linearly independent
3. $\text{rank}(\mathbf{A}) = N$
4. $\text{span}(\mathbf{A}) = \mathbb{R}^N$
5. If $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$
6. If $\mathbf{A}\mathbf{x} = \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$
7. For each $\mathbf{b} \in \mathbb{R}^N$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution
8. For each $\mathbf{b} \in \mathbb{R}^N$, the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution

All equivalent ways of saying that $T\mathbf{x} = \mathbf{A}\mathbf{x}$ is a bijection!

Example. For condition 5 the equivalence is

$$\text{if } \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y}, \text{ then } \mathbf{x} = \mathbf{y}$$

$$\iff \quad \text{if } T\mathbf{x} = T\mathbf{y}, \text{ then } \mathbf{x} = \mathbf{y}$$

$$\iff \quad T \text{ is one-to-one}$$

Since T is a linear map from $\mathbb{R}^N$ to $\mathbb{R}^N$,

$$\iff \quad T \text{ is a bijection}$$

Example. For condition 6 the equivalence is

$$\text{if } \mathbf{A}\mathbf{x} = \mathbf{0}, \text{ then } \mathbf{x} = \mathbf{0}$$

$$\iff \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\} = \{\mathbf{0}\}$$

$$\iff \{\mathbf{x} : T\mathbf{x} = \mathbf{0}\} = \{\mathbf{0}\}$$

$$\iff \ker(T) = \{\mathbf{0}\}$$

Since T is a linear map from $\mathbb{R}^N$ to $\mathbb{R}^N$,

$$\iff \quad T \text{ is a bijection}$$

Example. For condition 7 the equivalence is

for each $\mathbf{b} \in \mathbb{R}^N$, the equation $\mathbf{Ax} = \mathbf{b}$ has a solution

$\iff$ every $\mathbf{b} \in \mathbb{R}^N$ has an $\mathbf{x}$ such that $\mathbf{Ax} = \mathbf{b}$

$\iff$ every $\mathbf{b} \in \mathbb{R}^N$ has an $\mathbf{x}$ such that $T\mathbf{x} = \mathbf{b}$

$\iff$ T is onto

Since T is a linear map from $\mathbb{R}^N$ to $\mathbb{R}^N$,

$\iff$ T is a bijection

Now consider condition 2:

The columns of $\mathbf{A}$ are linearly independent

Let $\mathbf{e}_n$ be the n-th canonical basis vector in $\mathbb{R}^N$

Observe that $\mathbf{A}\mathbf{e}_n = \mathrm{col}_n(\mathbf{A})$

$$\therefore \quad T\mathbf{e}_n = \mathrm{col}_n(\mathbf{A})$$

$$\therefore \quad V := \{T\mathbf{e}_1, \ldots, T\mathbf{e}_N\} = \text{ columns of } \mathbf{A}$$

And V is linearly independent if and only if T is a bijection

Example. Consider a one good linear market system

$$q = a - bp \qquad \text{(demand)}$$

$$q = c + dp \qquad \text{(supply)}$$

Treating q and p as the unknowns, let's write in matrix form as

$$\begin{pmatrix} 1 & b \\ 1 & -d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

A unique solution exists whenever the columns are linearly independent

- means that $(b, -d)$ is not a scalar multiple of $\mathbf{1}$
- means that $b \neq -d$

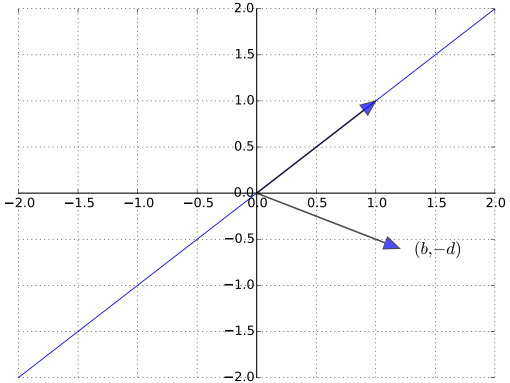


Figure : $(b, -d)$ is not a scalar multiple of $\mathbf{1}$

Example. Recall when we try to solve the system $\mathbf{Ax} = \mathbf{b}$ of this form

```
In [1]: import numpy as np
In [2]: from scipy.linalg import solve

In [3]: A = [[0, 2, 4],
   ...:       [1, 4, 8],
   ...:       [0, 3, 6]]

In [4]: b = (1, 2, 0)

In [5]: A, b = np.asarray(A), np.asarray(b)

In [6]: solve(A, b)
```

This is the output that we got

```
LinAlgError          Traceback (most recent call last)
<ipython-input-8-4fb5f41eaf7c> in <module>()
----> 1 solve(A, b)
/home/john/anaconda/lib/python2.7/site-packages/scipy/linal
     97             return x
     98         if info > 0:
---> 99             raise LinAlgError("singular matrix")
    100         raise ValueError('illegal value in %d-th argume
LinAlgError: singular matrix
```

The problem is that $\mathbf{A}$ is singular (not nonsingular)

- In particular, $\mathrm{col}_3(\mathbf{A}) = 2\,\mathrm{col}_2(\mathbf{A})$

Inverse Matrices

Given square matrix $\mathbf{A}$, suppose $\exists$ square matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}$$

Then

- $\mathbf{B}$ is called the **inverse** of $\mathbf{A}$, and written $\mathbf{A}^{-1}$
- $\mathbf{A}$ is called **invertible**

Fact. A square matrix $\mathbf{A}$ is nonsingular if and only if it is invertible

Remark

- $\mathbf{A}^{-1}$ is just the matrix corresponding to the linear map T^{-1}

Fact. Given nonsingular $N \times N$ matrix $\mathbf{A}$ and $\mathbf{b} \in \mathbb{R}^N$, the unique solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x}_b := \mathbf{A}^{-1}\mathbf{b}$$

Proof: Since $\mathbf{A}$ is nonsingular we already know any solution is unique

- T is a bijection, and hence one-to-one
- if $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{y} = \mathbf{b}$ then $\mathbf{x} = \mathbf{y}$

To show that $\mathbf{x}_b$ is indeed a solution we need to show that $\mathbf{A}\mathbf{x}_b = \mathbf{b}$

To see this, observe that

$$\mathbf{A}\mathbf{x}_b = \mathbf{A}\mathbf{A}^{-1}\mathbf{b} = \mathbf{I}\mathbf{b} = \mathbf{b}$$

Example. Recall the one good linear market system

$$\begin{aligned} q &= a - bp \\ q &= c + dp \end{aligned} \quad \iff \quad \begin{pmatrix} 1 & b \\ 1 & -d \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

Suppose that $a = 5$, $b = 2$, $c = 1$, $d = 1.5$

The matrix system is $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} := \begin{pmatrix} 1 & 2 \\ 1 & -1.5 \end{pmatrix}, \mathbf{x} := \begin{pmatrix} q \\ p \end{pmatrix}, \mathbf{b} := \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

Since $b \neq -d$ we can solve for the unique solution

Easy by hand but let's try on the computer

```
In [1]: import numpy as np
In [2]: from scipy.linalg import inv

In [3]: A = [[1, 2],
   ...:      [1, -1.5]]

In [4]: b = [5, 1]

In [5]: q, p = np.dot(inv(A), b)  # A^{-1} b

In [6]: q
Out[6]: 2.7142857142857144
In [7]: p
Out[7]: 1.1428571428571428
```

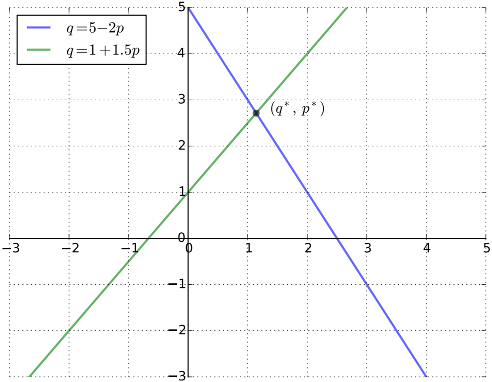


Figure : Equilibrium (p^*, q^*) in the one good case

Fact. In the 2×2 case, the inverse has the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 1 & -1.5 \end{pmatrix} \quad \Longrightarrow \quad \mathbf{A}^{-1} = \frac{1}{-3.5} \begin{pmatrix} -1.5 & -2 \\ -1 & 1 \end{pmatrix}$$

Example. Consider the N good linear demand system

$$q_n = \sum_{k=1}^{N} a_{nk} p_k + b_n, \quad n = 1, \ldots N \tag{1}$$

Task: take quantities $q_1, \ldots, q_N$ as given and find corresponding prices $p_1, \ldots, p_N$ — the "inverse demand curves"

We can write (1) as

$$\mathbf{q} = \mathbf{A}\mathbf{p} + \mathbf{b}$$

where vectors are N-vectors and $\mathbf{A}$ is $N \times N$

If the columns of $\mathbf{A}$ are linearly independent then a unique solution exists for each fixed $\mathbf{q}$ and $\mathbf{b}$, and is given by

$$\mathbf{p} = \mathbf{A}^{-1}(\mathbf{q} - \mathbf{b})$$

Left and Right Inverses

Given square matrix $\mathbf{A}$, a matrix $\mathbf{B}$ is called

- a **left inverse** of $\mathbf{A}$ if $\mathbf{BA} = \mathbf{I}$
- a **right inverse** of $\mathbf{A}$ if $\mathbf{AB} = \mathbf{I}$

By definition, a matrix that is both an left inverse and a right inverse is an inverse

Fact. If square matrix $\mathbf{B}$ is either a left or right inverse for $\mathbf{A}$, then $\mathbf{A}$ is nonsingular and $\mathbf{A}^{-1} = \mathbf{B}$

In other words, for square matrices,

$$\text{left inverse} \iff \text{right inverse} \iff \text{inverse}$$

Rules for Inverses

Fact. If $\mathbf{A}$ is nonsingular and $\alpha \neq 0$, then

1. $\mathbf{A}^{-1}$ is nonsingular and $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
2. $\alpha\mathbf{A}$ is nonsingular and $(\alpha\mathbf{A})^{-1} = \alpha^{-1}\mathbf{A}^{-1}$

Proof of part 2:

It suffices to show that $\alpha^{-1}\mathbf{A}^{-1}$ is the right inverse of $\alpha\mathbf{A}$

This is true because

$$\alpha\mathbf{A}\alpha^{-1}\mathbf{A}^{-1} = \alpha\alpha^{-1}\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

Fact. If $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ and nonsingular then

1. $\mathbf{AB}$ is also nonsingular
2. $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Proof I: Let T and U be the linear maps corresponding to $\mathbf{A}$ and $\mathbf{B}$

Recall that

- $T \circ U$ is the linear map corresponding to $\mathbf{AB}$
- Compositions of linear maps are linear
- Compositions of bijections are bijections

Hence $T \circ U$ is a linear bijection with $(T \circ U)^{-1} = U^{-1} \circ T^{-1}$

That is, $\mathbf{AB}$ is nonsingular with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$

Proof II:

A different proof that $\mathbf{AB}$ is nonsingular with inverse $\mathbf{B}^{-1}\mathbf{A}^{-1}$

Suffices to show that $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the right inverse of $\mathbf{AB}$

To see this, observe that

$$\mathbf{ABB}^{-1}\mathbf{A}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

Hence $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is a right inverse as claimed

When the Conditions Fail

Suppose as before we have

- an $N \times N$ matrix $\mathbf{A}$
- an $N \times 1$ vector $\mathbf{b}$

We seek a solution $\mathbf{x}$ to the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$

What if $\mathbf{A}$ is singular?

Then $T\mathbf{x} = \mathbf{A}\mathbf{x}$ is not a bijection, and in fact

- T cannot be onto (otherwise it's a bijection)
- T cannot be one-to-one (otherwise it's a bijection)

Hence neither existence nor uniqueness is guaranteed

Example. The matrix $\mathbf{A}$ with columns

$$\mathbf{a}_1 := \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}, \quad \mathbf{a}_2 := \begin{pmatrix} 3 \\ -4 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_3 := \begin{pmatrix} -3 \\ 4 \\ -1 \end{pmatrix}$$

is singular ($\mathbf{a}_3 = -\mathbf{a}_2$)

Its column space $\text{span}(\mathbf{A})$ is just a plane in $\mathbb{R}^2$

Recall $\mathbf{b} \in \text{span}(\mathbf{A})$

$$\iff \exists\, x_1, \ldots, x_N \text{ such that } \textstyle\sum_{k=1}^N x_k \operatorname{col}_k(\mathbf{A}) = \mathbf{b}$$

$$\iff \exists\, \mathbf{x} \text{ such that } \mathbf{A}\mathbf{x} = \mathbf{b}$$

Thus if $\mathbf{b}$ is not in this plane then $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solution

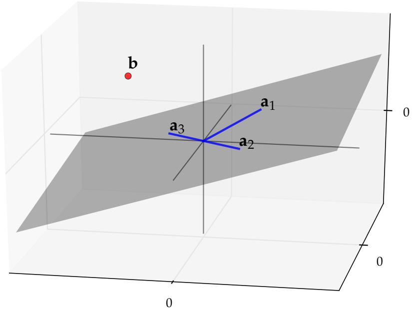


Figure : The vector $\mathbf{b}$ is not in $\text{span}(\mathbf{A})$

When $\mathbf{A}$ is $N \times N$ and singular how rare is scenario $\mathbf{b} \in \text{span}(\mathbf{A})$?

Answer: In a sense, very rare

We know that $\dim(\text{span}(\mathbf{A})) < N$

Such sets are always "very small" subset of $\mathbb{R}^N$ in terms of "volume"

- A $K < N$ dimensional subspace has "measure zero" in $\mathbb{R}^N$
- A "randomly chosen" $\mathbf{b}$ has zero probability of being in such a set

Example. Consider the case where $N = 3$ and $K = 2$

A two-dimensional linear subspace is a 2D plane in $\mathbb{R}^3$

This set has no volume because planes have no "thickness"

All this means that if $\mathbf{A}$ is singular then existence of a solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ typically fails

In fact the problem is worse — uniqueness fails as well

Fact. If $\mathbf{A}$ is a singular matrix and $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution then it has an infinity (in fact a continuum) of solutions

Proof: Let $\mathbf{A}$ be singular and let $\mathbf{x}$ be a solution

Since $\mathbf{A}$ is singular there exists a nonzero $\mathbf{y}$ with $\mathbf{A}\mathbf{y} = \mathbf{0}$

But then $\alpha\mathbf{y} + \mathbf{x}$ is also a solution for any $\alpha \in \mathbb{R}$ because

$$\mathbf{A}(\alpha\mathbf{y} + \mathbf{x}) = \alpha\mathbf{A}\mathbf{y} + \mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x} = \mathbf{b}$$

Determinants

Let $S(N)$ be set of all bijections from $\{1, \ldots, N\}$ to itself

For $\pi \in S(N)$ we define the **signature** of π as

$$\operatorname{sgn}(\pi) := \prod_{m<n} \frac{\pi(m) - \pi(n)}{m - n}$$

The **determinant** of $N \times N$ matrix $\mathbf{A}$ is then given as

$$\det(\mathbf{A}) := \sum_{\pi \in S(N)} \operatorname{sgn}(\pi) \prod_{n=1}^{N} a_{\pi(n)n}$$

- You don't need to understand or remember this for our course

Fact. In the $N = 2$ case this definition reduces to

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

- Remark: But you do need to remember this 2×2 case

Example

$$\det \begin{pmatrix} 2 & 0 \\ 7 & -1 \end{pmatrix} = (2 \times -1) - (7 \times 0) = -2$$

Important facts concerning the determinant

Fact. If $\mathbf{I}$ is the $N \times N$ identity, $\mathbf{A}$ and $\mathbf{B}$ are $N \times N$ matrices and $\alpha \in \mathbb{R}$, then

1. $\det(\mathbf{I}) = 1$
2. $\mathbf{A}$ is nonsingular if and only if $\det(\mathbf{A}) \neq 0$
3. $\det(\mathbf{AB}) = \det(\mathbf{A}) \det(\mathbf{B})$
4. $\det(\alpha \mathbf{A}) = \alpha^N \det(\mathbf{A})$
5. $\det(\mathbf{A}^{-1}) = (\det(\mathbf{A}))^{-1}$

Example. Thus singularity in the 2×2 case is equivalent to

$$\det(\mathbf{A}) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} = 0$$

Ex. Let $\mathbf{a}_i := \mathrm{col}_i(\mathbf{A})$ and assume that $a_{ij} \neq 0$ for each i, j

Show the following are equivalent:

1. $a_{11}a_{22} = a_{12}a_{21}$
2. $\mathbf{a}_1 = \lambda \mathbf{a}_2$ for some $\lambda \in \mathbb{R}$

```
In [1]: import numpy as np

In [2]: A = np.random.randn(2, 2)  # Random matrix

In [3]: A
Out[3]:
array([[-0.70120551,  0.57088203],
       [ 0.40757074, -0.72769741]])

In [4]: np.linalg.det(A)
Out[4]: 0.27759063032043652

In [5]: 1.0 / np.linalg.det(np.linalg.inv(A))
Out[5]: 0.27759063032043652
```

As an exercise, let's now show that any right inverse is an inverse

Fix square $\mathbf{A}$ and suppose $\mathbf{B}$ is a right inverse:

$$\mathbf{A}\mathbf{B} = \mathbf{I} \tag{2}$$

Applying the determinant to both sides gives $\det(\mathbf{A}) \det(\mathbf{B}) = 1$

Hence $\mathbf{B}$ is nonsingular (why?) and we can

1. multiply (2) by $\mathbf{B}$ to get $\mathbf{B}\mathbf{A}\mathbf{B} = \mathbf{B}$
2. then postmultiply by $\mathbf{B}^{-1}$ to get $\mathbf{B}\mathbf{A} = \mathbf{I}$

We see that $\mathbf{B}$ is also left inverse, and therefore an inverse of $\mathbf{A}$

Ex. Do the left inverse case

Other Linear Equations

So far we have considered the nice $N \times N$ case for equations

- number of equations $=$ number of unknowns

We have to deal with other cases too

Underdetermined systems:

- eqs $<$ unknowns

Overdetermined systems:

- eqs $>$ unknowns

Overdetermined Systems

Consider the system $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A}$ is $N \times K$ and $K < N$

- The elements of $\mathbf{x}$ are the unknowns
- More equations than unknowns $(N > K)$

May not be able to find an $\mathbf{x}$ that satisfies all N equations

Let's look at this in more detail...

Fix $N \times K$ matrix $\mathbf{A}$ with $K < N$

Let $T \colon \mathbb{R}^K \to \mathbb{R}^N$ be defined by $T\mathbf{x} = \mathbf{A}\mathbf{x}$

We know these to be equivalent:

1. there exists an $\mathbf{x} \in \mathbb{R}^K$ with $\mathbf{A}\mathbf{x} = \mathbf{b}$
2. $\mathbf{b}$ has a preimage under T
3. $\mathbf{b}$ is in $\mathrm{rng}(T)$
4. $\mathbf{b}$ is in $\mathrm{span}(\mathbf{A})$

We also know T cannot be onto (maps small to big space)

Hence $\mathbf{b} \in \mathrm{span}(\mathbf{A})$ will not always hold

Given our assumption that $K < N$, how rare is the scenario $\mathbf{b} \in \text{span}(\mathbf{A})$?

Answer: We talked about this before — it's very rare

We know that $\dim(\text{rng}(T)) = \dim(\text{span}(\mathbf{A})) \leq K < N$

A $K < N$ dimensional subspace has "measure zero" in $\mathbb{R}^N$

So should we give up on solving $\mathbf{A}\mathbf{x} = \mathbf{b}$ in the overdetermined case?

What's typically done is we try to find a best approximation

To define "best" we need a way of ranking approximations

The standard way is in terms of Euclidean norm

In particular, we search for the $\mathbf{x}$ that solves

$$\min_{\mathbf{x} \in \mathbb{R}^K} \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$$

Details later

Underdetermined Systems

Now consider $\mathbf{A}\mathbf{x} = \mathbf{b}$ when $\mathbf{A}$ is $N \times K$ and $K > N$

Let $T \colon \mathbb{R}^K \to \mathbb{R}^N$ be defined by $T\mathbf{x} = \mathbf{A}\mathbf{x}$

Now T maps from a larger to a smaller place

This tells us that T is not one-to-one

Hence solutions are not in general unique

In fact the following is true

Ex. Show that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution and $K > N$, then the same equation has an infinity of solutions

Remark: Working with underdetermined systems is relatively rare in economics / elsewhere