ECON2125/8013

Lecture 5

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Semester 1, 2015

Announcements

- New tutorial has opened

Tuples

We often organize collections with natural order into "tuples"

A **tuple** is

- a finite sequence of terms
- denoted using notation such as (a_1, a_2) or (x_1, x_2, x_3)

Example. Flip a coin 10 times and let

- 0 represent tails and 1 represent heads

Typical outcome $(1,1,0,0,0,0,1,0,1,1)$

Generic outcome $(b_1, b_2, \ldots, b_{10})$ for $b_n \in \{0,1\}$

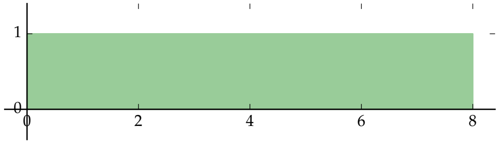
Cartesian Products

We make collections of tuples using Cartesian products

The **Cartesian product** of $A_1, \ldots, A_N$ is the set

$$A_1 \times \cdots \times A_N := \{(a_1, \ldots, a_N) : a_n \in A_n \text{ for } n = 1, \ldots, N\}$$

Example. $[0, 8] \times [0, 1] = \{(x_1, x_2) : 0 \leq x_1 \leq 8, 0 \leq x_2 \leq 1\}$



Example. Set of all outcomes from flip experiment is

$$B := \{(b_1, \ldots, b_{10}) : b_n \in \{0,1\} \text{ for } n = 1, \ldots, 10\}$$

$$= \{0,1\} \times \cdots \times \{0,1\} \quad (10 \text{ products})$$

Example. The **vector space** $\mathbb{R}^N$ is the Cartesian product

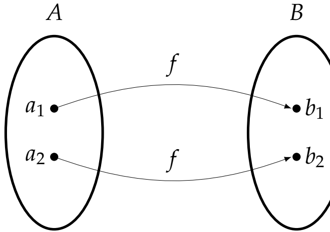
$$\mathbb{R}^N = \mathbb{R} \times \cdots \times \mathbb{R} \quad (N \text{ times})$$

$$= \{\text{all tuples } (x_1, \ldots, x_N) \text{ with } x_n \in \mathbb{R}\}$$

Functions

A **function** f from set A to set B is a rule that associates to each element of A a uniquely determined element of B

- $f \colon A \to B$ means that f is a function from A to B



A is called the **domain** of f and B is called the **codomain**

Example. f defined by

$$f(x) = \exp(-x^2)$$

is a function from $\mathbb{R}$ to $\mathbb{R}$

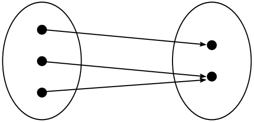
Sometimes we write the whole thing like this

$$f\colon \mathbb{R} \to \mathbb{R}$$

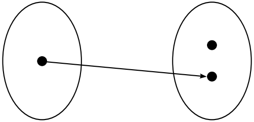
$$x \mapsto \exp(-x^2)$$

or this

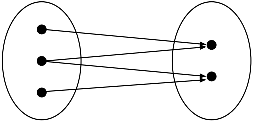
$$f\colon \mathbb{R} \ni x \mapsto \exp(-x^2) \in \mathbb{R}$$



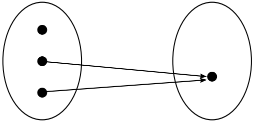
A function



A function

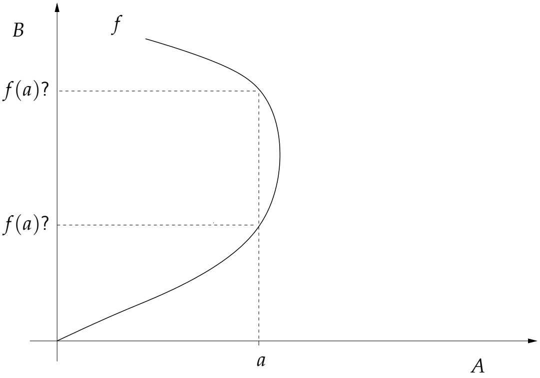


Not a function

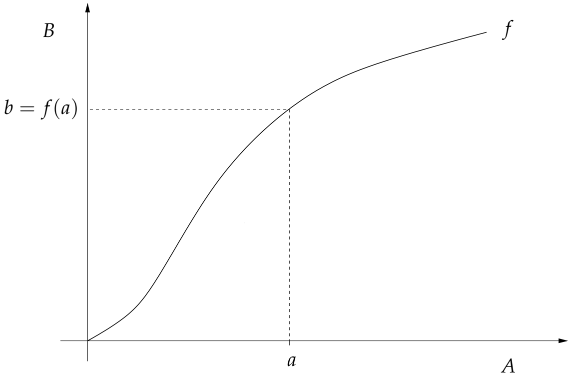


Not a function

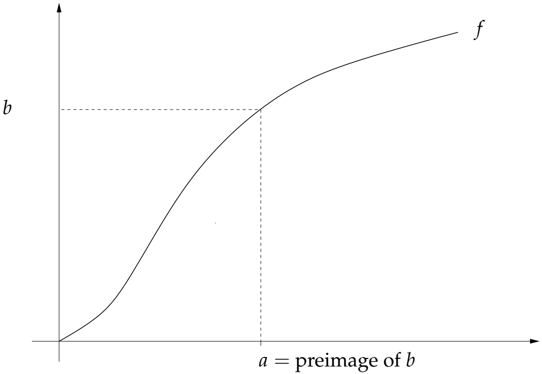
Not a function



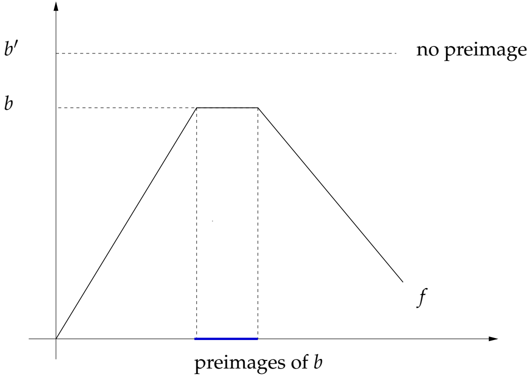
For each $a \in A$, $f(a) \in B$ is called the **image of a under f**



If $f(a) = b$ then a is called a **preimage of b under f**



A point in B can have one, many or zero preimages



The codomain of a function is not uniquely pinned down

Example. Consider the mapping defined by

$$f(x) = \exp(-x^2)$$

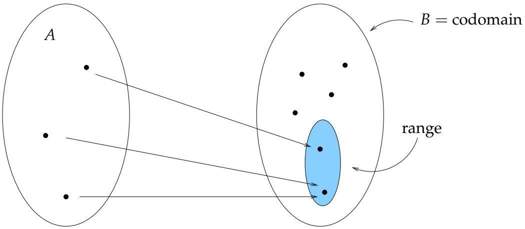
Both of these statements are valid:

- f a function from $\mathbb{R}$ to $\mathbb{R}$
- f a function from $\mathbb{R}$ to $(0, \infty)$

The smallest possible codomain is called the range – next slide

The **range** of $f\colon A \to B$ is the set

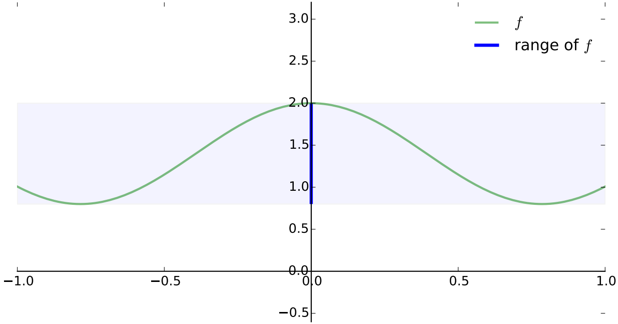
$$\mathrm{rng}(f) := \{b \in B : f(a) = b \text{ for some } a \in A\}$$



Example. Let $f\colon [-1,1] \to \mathbb{R}$ be defined by

$$f(x) = 0.6\cos(4x) + 1.4$$

Then $\operatorname{rng}(f) = [0.8, 2.0]$



Example. If $f\colon [0,1] \to \mathbb{R}$ is defined by

$$f(x) = 2x$$

then $\operatorname{rng}(f) = [0,2]$

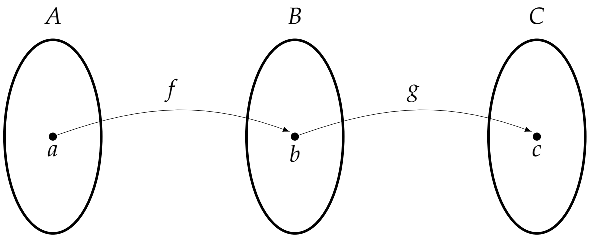
Example. If $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \exp(x)$$

then $\operatorname{rng}(f) = (0,\infty)$

The **composition** of $f\colon A \to B$ and $g\colon B \to C$ is the function $g \circ f$ from A to C defined by

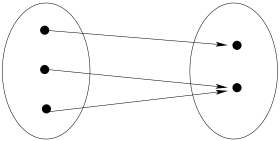
$$(g \circ f)(a) = g(f(a)) \qquad (a \in A)$$



Onto Functions

A function $f\colon A \to B$ is called **onto** if every element of B is the image under f of at least one point in A.

Equivalently, $\operatorname{rng}(f) = B$



Fact. $f\colon A \to B$ is onto if and only if each element of B has at least one preimage under f

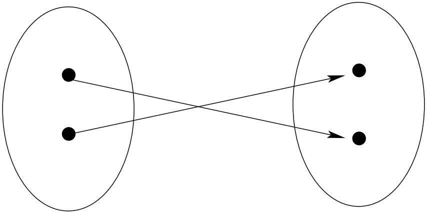


Figure : Onto

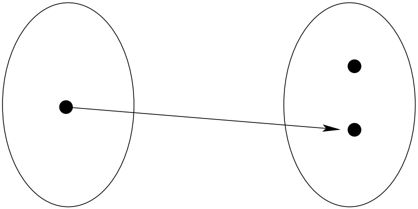


Figure : Not onto

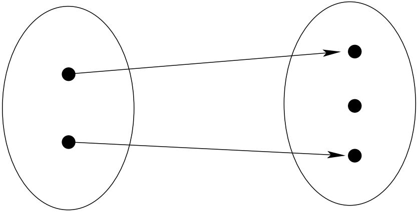


Figure : Not onto

Example. The function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = ax^3 + bx^2 + cx + d$$

is onto whenever $a \neq 0$

To see this pick any $y \in \mathbb{R}$

We claim $\exists\, x \in \mathbb{R}$ such that $f(x) = y$

Equivalent:

$$\exists\, x \in \mathbb{R} \ \text{ s.t. } \ ax^3 + bx^2 + cx + d - y = 0$$

Fact. Every cubic equation with $a \neq 0$ has at least one real root

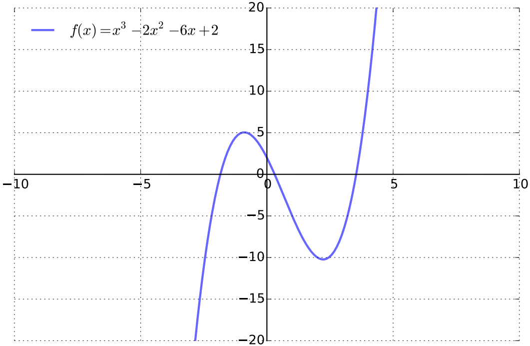


Figure : Cubic functions from $\mathbb{R}$ to $\mathbb{R}$ are always onto

One-to-One Functions

A function $f: A \to B$ is called **one-to-one** if distinct elements of A are always mapped into distinct elements of B.

That is, f is one-to-one if

$$a \in A,\ a' \in A \text{ and } a \neq a' \implies f(a) \neq f(a')$$

Equivalently,

$$f(a) = f(a') \implies a = a'$$

Fact. $f: A \to B$ is one-to-one if and only if each element of B has at most one preimage under f

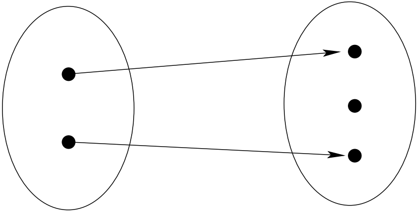


Figure : One-to-one

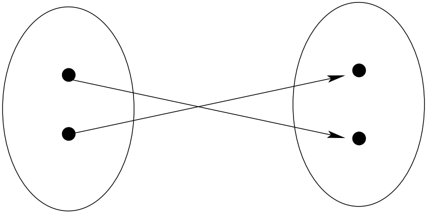


Figure : One-to-one

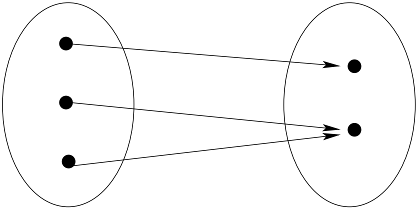


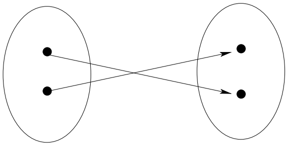
Figure : Not one-to-one

Bijections

A function that is

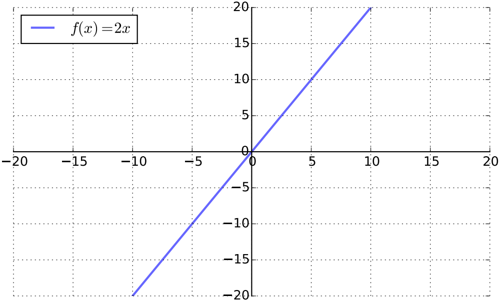
1. one-to-one and
2. onto

is called a **bijection** or **one-to-one correspondence**

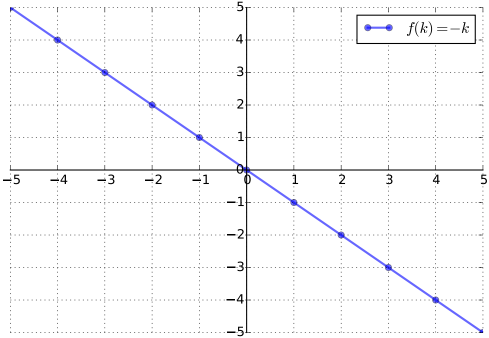


Fact. $f\colon A \to B$ is a bijection if and only if each $b \in B$ has one and only one preimage in A

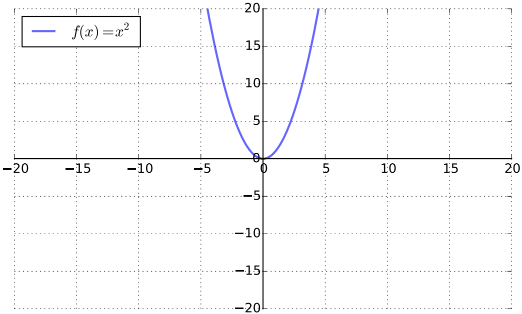
Example. $x \mapsto 2x$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$



Example. $k \mapsto -k$ is a bijection from $\mathbb{Z}$ to $\mathbb{Z}$



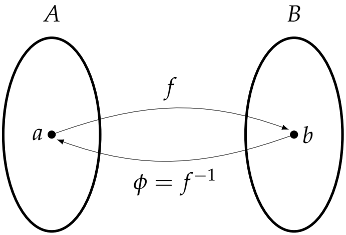
Example. $x \mapsto x^2$ is not a bijection from $\mathbb{R}$ to $\mathbb{R}$



Fact. If $f\colon A \to B$ a bijection, then there exists a unique function $\phi\colon B \to A$ such that

$$\phi(f(a)) = a, \qquad \forall\, a \in A$$

That function ϕ is called the **inverse** of f and written f^{-1}



Example. Let

- $f\colon \mathbb{R} \to (0,\infty)$ be defined by $f(x) = \exp(x) := e^x$
- $\phi\colon (0,\infty) \to \mathbb{R}$ be defined by $\phi(x) = \log(x)$

Then $\phi = f^{-1}$ because, for any $a \in \mathbb{R}$,

$$\phi(f(a)) = \log(\exp(a)) = a$$

Fact. If $f\colon A \to B$ is one-to-one, then $f\colon A \to \operatorname{rng}(f)$ is a bijection

Fact. Let $f\colon A \to B$ and $g\colon B \to C$ be bijections

1. f^{-1} is a bijection and its inverse is f
2. $f^{-1}(f(a)) = a$ for all $a \in A$
3. $f(f^{-1}(b)) = b$ for all $b \in B$
4. $g \circ f$ is a bijection from A to C and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

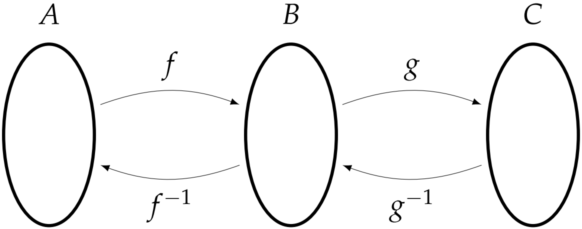


Illustration of $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$

Counting

Counting methods answer common questions such as

- How many arrangements of a sequence?
- How many subsets of a set?

They also address deeper problems such as

- How “large” is a given set?
- Can we compare size of sets even when they are infinite?

Counting Finite Sequences

The key rule is: multiply possibilities

Example. Can travel from Sydney to Tokyo in 3 ways and Tokyo to NYC in 8 ways $\implies$ can travel from Sydney to NYC in 24 ways

Example. Number of 10 letter passwords from the lowercase letters `a, b, ..., z` is

$$26^{10} = 141,167,095,653,376$$

Example. Number of possible distinct outcomes (i, j) from 2 rolls of a dice is

$$6 \times 6 = 36$$

Counting Cartesian Products

Fact. If A_n are finite for $n = 1, \dots, N$, then

$$\#(A_1 \times \cdots \times A_N) = (\#A_1) \times \cdots \times (\#A_N)$$

That is, number of possible tuples = product of the number of possibilities for each element

Example. Number of binary sequences of length 10 is

$$\#[\{0,1\} \times \cdots \times \{0,1\}] = 2 \times \cdots \times 2 = 2^{10}$$

Cardinality

If a bijection exists between sets A and B they are said to have the **same cardinality**, and we write $|A| = |B|$

Fact. If $|A| = |B|$ and A and B are finite then A and B have the same number of elements

Ex. Convince yourself this is true

Hence "same cardinality" is analogous to "same size"

- But cardinality applies to infinite sets as well

Fact. If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$

Proof:

- Since $|A| = |B|$, exists a bijection $f\colon A \to B$
- Since $|B| = |C|$, exists a bijection $g\colon B \to C$

Let $h := g \circ f$

Then h is a bijection from A to C

Hence $|A| = |C|$

A nonempty set A is called **finite** if

$$|A| = |\{1, 2, \ldots, n\}| \quad \text{for some} \quad n \in \mathbb{N}$$

Otherwise called **infinite**

Sets that either

1. are finite, or
2. have the same cardinality as $\mathbb{N}$

are called **countable**

- write $|A| = \aleph_0$

Example. $-\mathbb{N} := \{\ldots, -4, -3, -2, -1\}$ is countable

$$\begin{array}{ccc} -1 & \leftrightarrow & 1 \\ -2 & \leftrightarrow & 2 \\ -3 & \leftrightarrow & 3 \\ & \vdots & \\ -n & \leftrightarrow & n \\ & \vdots & \end{array}$$

Formally: $f(k) = -k$ is a bijection from $-\mathbb{N}$ to $\mathbb{N}$

Example. $E := \{2, 4, \ldots\}$ is countable

$$\begin{array}{ccc} 2 & \leftrightarrow & 1 \\ 4 & \leftrightarrow & 2 \\ 6 & \leftrightarrow & 3 \\ & \vdots & \\ 2n & \leftrightarrow & n \\ & \vdots & \end{array}$$

Formally: $f(k) = k/2$ is a bijection from E to $\mathbb{N}$

Example. $\{100, 200, 300, \ldots\}$ is countable

$$
\begin{array}{ccc}
100 & \leftrightarrow & 1 \\
200 & \leftrightarrow & 2 \\
300 & \leftrightarrow & 3 \\
 & \vdots & \\
100n & \leftrightarrow & n \\
 & \vdots &
\end{array}
$$

Fact. Nonempty subsets of countable sets are countable

Fact. Finite unions of countable sets are countable

Sketch of proof, for

- A and B countable $\implies$ $A \cup B$ countable
- A and B are disjoint and infinite

By assumption, can write $A = \{a_1, a_2, \ldots\}$ and $B = \{b_1, b_2, \ldots\}$

Now count it like so:

$$\begin{array}{ccccccccc}
a_1 & & a_2 & & a_3 & & a_4 & \cdots \\
\downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow \\
b_1 & & b_2 & & b_3 & & b_4 & \cdots
\end{array}$$

Example. $\mathbb{Z} = \{\ldots, -2, -1\} \cup \{0\} \cup \{1, 2, \ldots\}$ is countable

Fact. Finite Cartesian products of countables are countable

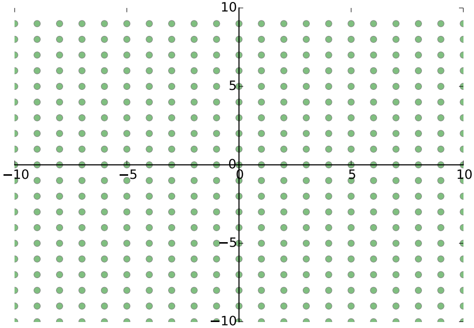
Sketch of proof, for

- A and B countable $\implies A \times B$ countable
- A and B are disjoint and infinite

Now count like so:

$$
\begin{array}{ccccccc}
(a_1, b_1) & \rightarrow & (a_1, b_2) & & (a_1, b_3) & \rightarrow \cdots \\
& \swarrow & & \nearrow & & \\
(a_2, b_1) & & (a_2, b_2) & & \cdots & \\
\downarrow & \nearrow & & & & \\
(a_3, b_1) & & \vdots & & & \\
\vdots & & & & &
\end{array}
$$

Example. $\mathbb{Z} \times \mathbb{Z} = \{(p, q) : p \in \mathbb{Z}, q \in \mathbb{Z}\}$ is countable



Fact. $\mathbb{Q}$ is countable

Proof: By definition

$$\mathbb{Q} := \left\{ \text{all } \frac{p}{q} \text{ where } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \right\}$$

Consider the function ϕ defined by $\phi(p/q) = (p, q)$

- A one-to-one function from $\mathbb{Q}$ to $\mathbb{Z} \times \mathbb{N}$
- A bijection from $\mathbb{Q}$ to $\operatorname{rng}(\phi)$

Since $\mathbb{Z} \times \mathbb{N}$ is countable, so is $\operatorname{rng}(\phi) \subset \mathbb{Z} \times \mathbb{N}$

Hence $\mathbb{Q}$ is also countable

An example of an uncountable set is all binary sequences

$$\{0,1\}^{\mathbb{N}} := \{(b_1, b_2, \ldots) : b_n \in \{0,1\} \text{ for each } n\}$$

Sketch of proof: If this set were countable then it could be listed as follows:

$$\begin{array}{ccl} 1 & \leftrightarrow & \color{red}{a_1}, a_2, a_3, a_4, \ldots \\ 2 & \leftrightarrow & b_1, \color{red}{b_2}, b_3, b_4, \ldots \\ 3 & \leftrightarrow & c_1, c_2, \color{red}{c_3}, c_4, \ldots \\ 4 & \leftrightarrow & d_1, d_2, d_3, \color{red}{d_4}, \ldots \\ \vdots & & \qquad \vdots \end{array}$$

Such a list is never complete: Cantor's diagonalization argument

Cardinality of $\{0,1\}^{\mathbb{N}}$ called the **power of the continuum**

Other sets with the power of the continuum

- $\mathbb{R}$
- (a, b) for any $a < b$
- $[a, b]$ for any $a < b$
- $\mathbb{R}^N$ for any finite $N \in \mathbb{N}$

Continuum hypothesis: Every nonempty subset of $\mathbb{R}$ is either countable or has the power of the continuum

- Not a homework exercise!

Example. $\mathbb{R}$ and $(-1,1)$ have the same cardinality because $x \mapsto 2\arctan(x)/\pi$ is a bijection from $\mathbb{R}$ to $(-1,1)$

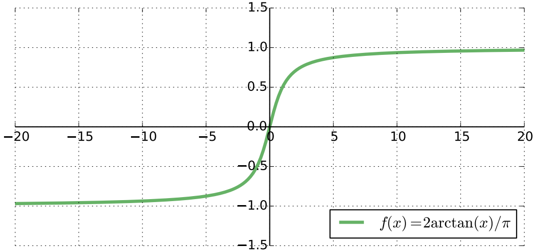


Figure : Same cardinality