ECON2125/4021/8013

Lecture 26

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Semester 1, 2015

Today's Lecture

Last lecture: Review of probability, analysis and dynamics

Comments on the exam

- Same format as midterm but 3 hours instead of 2
    - average difficulty of questions will be lower
    - but you'll still need to study hard
- See comments on the midterm (start of lecture 14) for hints on how to solve problems
- Practice questions (solved exercises) are very important

All aspects of the course will be covered

- optimization problems
- linear algebra
- probability
- basic analysis and its applications
- dynamics

Further announcement:

- My June 1st 9-11am office hours will be shifted to June 2nd same time
- Qingyin and Guanlong will run their usual consultation sessions on June 5th

Probability Review

Key Idea. The law of total probability can help us break problems down into simple parts

$$\mathbb{P}(A) = \mathbb{P}(A \mid B_1)\mathbb{P}(B_1) + \mathbb{P}(A \mid B_2)\mathbb{P}(B_2)$$

Exercise: Let's use this law to predict unemployment outcomes

We will take as given

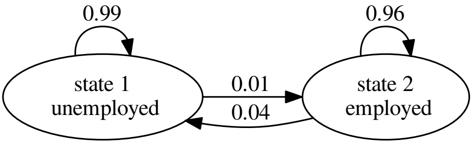
- the current unemployment rate
- the transition probabilities for workers

Employment Dynamics

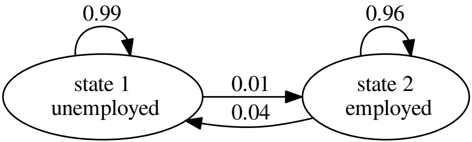
Consider a worker who can be either employed or unemployed in each period

Suppose further that

- When unemployed, finds a job next period with 1% probability
- When employed, loses her job next period with 4% probability



If we look at the graph again we see 4 transition probabilities



1. $\pi_{11} := \mathbb{P}\{\text{state 1} \to \text{state 1}\} = 0.99$
2. $\pi_{12} := \mathbb{P}\{\text{state 1} \to \text{state 2}\} = 0.01$
3. $\pi_{22} := \mathbb{P}\{\text{state 2} \to \text{state 2}\} = 0.96$
4. $\pi_{21} := \mathbb{P}\{\text{state 2} \to \text{state 1}\} = 0.04$

Suppose that the current unemployment rate is 8%

Hence if we pick a worker randomly, then $\mathbb{P}\{\text{state } 1 = 0.08\}$

What is her probability of being unemployed next period?

By the law of total probability,

$$\mathbb{P}\{\text{state 1 at } t+1\} =$$

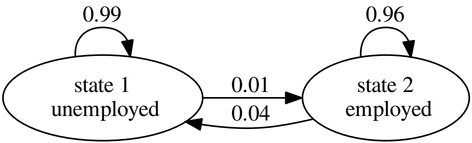
$$\mathbb{P}\{\text{state 1 at } t+1 \mid \text{state 1 at } t\}\mathbb{P}\{\text{state 1 at } t\}$$

$$+\,\mathbb{P}\{\text{state 1 at } t+1 \mid \text{state 2 at } t\}\mathbb{P}\{\text{state 2 at } t\}$$

or

$$\mathbb{P}\{\text{state 1 at } t+1\} = \pi_{11}\mathbb{P}\{\text{state 1 at } t\} + \pi_{21}\mathbb{P}\{\text{state 2 at } t\}$$

Let's fill in the numbers by referring back to the graph



We see that

$$\mathbb{P}\{\text{state 1 at } t+1\} = \pi_{11}\mathbb{P}\{\text{state 1 at } t\} + \pi_{21}\mathbb{P}\{\text{state 2 at } t\}$$

$$= 0.99 \times 0.08 + 0.04 \times 0.92 = 0.116$$

The general rule is

$$\mathbb{P}\{\text{state } j \text{ at } t+1\} = \pi_{1j}\mathbb{P}\{\text{state 1 at } t\} + \pi_{2j}\mathbb{P}\{\text{state 2 at } t\}$$

We simplify notation by letting

$$\mathbf{p}_t := \begin{pmatrix} \mathbb{P}\{\text{state 1 at } t\} \\ \mathbb{P}\{\text{state 2 at } t\} \end{pmatrix} \quad \text{and} \quad \mathbf{\Pi} := \begin{pmatrix} \pi_{11} & \pi_{21} \\ \pi_{12} & \pi_{22} \end{pmatrix}$$

Ex. Show that the two equations ($j = 1, 2$) at the top of the slide can be expressed as

$$\mathbf{p}_{t+1} = \mathbf{\Pi}\mathbf{p}_t$$

Let's make sure that this gives us the same result

We said a random worker has an 8% chance of being unemployed:

$$\mathbf{p}_t = \begin{pmatrix} 0.08 \\ 0.92 \end{pmatrix}$$

We then update to next period probabilities by

$$\mathbf{p}_{t+1} = \mathbf{\Pi} \mathbf{p}_t = \begin{pmatrix} 0.99 & 0.04 \\ 0.01 & 0.96 \end{pmatrix} \begin{pmatrix} 0.08 \\ 0.92 \end{pmatrix} = \begin{pmatrix} 0.116 \\ 0.884 \end{pmatrix}$$

As before, the chance of being unemployed next period $= 11.6\%$

Our matrix expression allows us to compute other useful values

Example. We can repeat the method of obtaining next period probabilities, but now starting at

$$\mathbf{p}_{t+1} = \begin{pmatrix} 0.116 \\ 0.884 \end{pmatrix}$$

Updating in the same way gives

$$\mathbf{p}_{t+2} = \mathbf{\Pi}\mathbf{p}_{t+1} = \begin{pmatrix} 0.99 & 0.04 \\ 0.01 & 0.96 \end{pmatrix} \begin{pmatrix} 0.116 \\ 0.884 \end{pmatrix} = \begin{pmatrix} 0.150 \\ 0.850 \end{pmatrix}$$

Thus, the worker has a 15% chance of being unemployed at $t + 2$

Review of Expectations

The three most important properties of expectations are

1. Linearity:
$$\mathbb{E}\left[\sum_{n=1}^{N} \alpha_n X_n\right] = \sum_{n=1}^{N} \alpha_n \mathbb{E}\left[X_n\right]$$
2. Monotonicity:
$$X \le Y \quad \Longrightarrow \quad \mathbb{E}\left[X\right] \le \mathbb{E}\left[Y\right]$$
3. Expectation of an indicator function is the prob of the event:
$$\mathbb{E}\left[\mathbb{1}\{A \text{ occurs}\}\right] = \mathbb{P}(A)$$

Review of the Law of Large Numbers

The LLN tells us that if $\{X_n\}$ is IID and $h(X_1)$ has finite expectation, then

$$\frac{1}{N}\sum_{n=1}^{N} h(X_n) \stackrel{p}{\to} \mathbb{E}\left[h(X_1)\right] \quad \text{as} \quad N \to \infty \qquad (\star)$$

The LLN applies to probabilities as well

Example. If we take $h(x) = \mathbb{1}\{x \in B\}$ then $(\star)$ gives

$$\frac{1}{N}\sum_{n=1}^{N} \mathbb{1}\{X_n \in B\} \stackrel{p}{\to} \mathbb{E}\left[\mathbb{1}\{X_n \in B\}\right] = \mathbb{P}\{X_n \in B\}$$

In words: The fraction of times we observe a given outcome is close to its probability in large samples

Example. In our worker example, suppose again that the unemployment rate is 8%

- $\implies$ our randomly sampled worker has a 11.6% prob of unemployment next period

Suppose now that

- worker outcomes are independent
- all workers have the same transition probabilities

Then each worker has an 11.6% percent chance of being unemployed next period

By the LLN, in a large population this will be close to the fraction of unemployed — the unemployment rate

Review of Analysis

Analysis concerns concepts such as

- open sets, closed sets, bounded sets
- continuous functions, increasing functions
- derivatives and limits

Why do we need analysis?

In essence because algebra, while incredibly powerful, has limits

We easily run into situations where we can't solve things with pencil and paper

- Need to refer to general theorems in that case

Solving Nonlinear Equations

Much of analysis is about solving equations such as

$$f(x) = y$$

Given f and y, what is the x?

- Does it exist? Is it unique? How can we find it?

Key Idea. We can solve nonlinear equations by turning them into either fixed point or zero (root) finding problems, and then applying

- the Intermediate Value Theorem
- Brouwer's fixed point theorem
- Banach's contraction mapping theorem, etc.

Example. If $X \sim N(0,1)$, what is the x s.t. $\mathbb{P}\{X \leq x\} = 0.9$?

We can convert this into a zero finding problem by setting

$$g(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt - 0.9$$

What is the x such that $g(x) = 0$?

There's no neat solution for the integral, but we can be sure that a solution exists by the Intermediate Value Theorem

Ex. Using the **Facts** in lectures 12 and 13, argue that

- g is continuous
- $g(a) < 0$ for sufficiently small a
- $g(b) > 0$ for sufficiently large b

Review of Dynamical Systems

Difference equations such as

$$\mathbf{x}_{t+1} = g(\mathbf{x}_t)$$

are best thought about in the language of dynamical systems

Recall that a dynamical system is a pair (S, g) where

- S is any nonempty set
- g is a function from S back to itself

We always have to make sure that

1. We clarify what S is
2. We verify that $g \colon S \to S$ is valid

Example. In our worker application, state probabilities were updated by the rule

$$\mathbf{p}_{t+1} = \mathbf{\Pi} \mathbf{p}_t$$

We can view this as a dynamical system $(\mathbb{D}, g)$, where

$$\mathbb{D} = \{\mathbf{p} \in \mathbb{R}^2 : \mathbf{p} \geq \mathbf{0}, \; \mathbf{1}'\mathbf{p} = 1\}$$

$$g(\mathbf{p}) = \mathbf{\Pi} \mathbf{p}$$

Here $\mathbf{1}$ is a 2×1 vector of ones

Thus, $\mathbb{D}$ is all vectors with nonnegative elements that sum to 1

- Often called the "unit simplex"

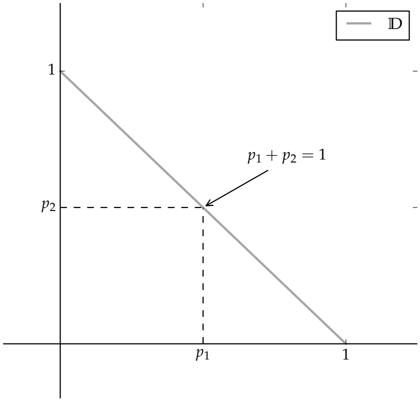


Figure : The unit simplex in $\mathbb{R}^2$

But is $\mathbb{D}$ a valid state space, in the sense that $g \colon \mathbb{D} \to \mathbb{D}$?

Fact. If $\mathbf{p} \in \mathbb{D}$, then $\mathbf{\Pi p} \in \mathbb{D}$

Proof: Let $\mathbf{p}$ and $\mathbf{\Pi}$ be as in the statement above

We need to check that $\mathbf{\Pi p} \geq \mathbf{0}$ and $\mathbf{1}' \mathbf{\Pi p} = 1$

Since all terms in $\mathbf{p}$ and $\mathbf{\Pi}$ are nonnegative, so are those in $\mathbf{\Pi p}$

Ex. Show that $\mathbf{1}' \mathbf{\Pi} = \mathbf{1}'$

It follows that

$$\mathbf{1}' \mathbf{\Pi p} = \mathbf{1}' \mathbf{p} = 1$$

Hence $\mathbf{\Pi p} \in \mathbb{D}$ as claimed

Trajectories

Returning to the dynamical system (S, g), recall that trajectories are sequences of the form

$$\mathbf{x}_t = g^t(\mathbf{x}_0), \qquad \mathbf{x}_0 = \text{given initial condition}$$

In general they can be

- monotone
- periodic
- convergent
- chaotic, etc.

Example. The unemployment distributions were found to obey

$$\mathbf{p}_{t+1} = \mathbf{\Pi}\mathbf{p}_t$$

We can view these are trajectories of the dynamical system $(\mathbb{D}, g)$

They can also be written as

$$\mathbf{p}_t = g^t(\mathbf{p}_0) = \mathbf{\Pi}^t\mathbf{p}_0$$

Let's look at a picture of this trajectory from initial condition

$$\mathbf{p}_0 = \begin{pmatrix} 0.08 \\ 0.92 \end{pmatrix}$$

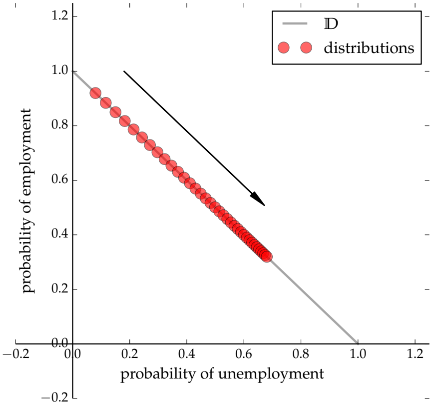


Figure : The sequence $\mathbf{p}_t = \mathbf{\Pi}^t \mathbf{p}_0$ when $\mathbf{p}_0' = (0.08 \;\; 0.92)$

Steady States

Let's go back to a general dynamical system (S, g)

One topic of interest is existence / uniqueness of steady states

A steady state of (S, g) is just a fixed point of g in S

Key Idea. To study steady states we can use fixed point theory

- Brouwer's fixed point theorem to get existence, or
- The contraction mapping theorem or Neumann series lemma to get
    1. existence
    2. uniqueness
    3. global stability

Example. Linear systems have the form

$$g(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{b}$$

where $\mathbf{A}$ is $N \times N$ and $\mathbf{b}$ is $N \times 1$

Key Idea. If $\rho(\mathbf{A}) < 1$, then $(\mathbb{R}^N, g)$ is globally stable, with unique steady state

$$\mathbf{x}^* = \sum_{i=0}^{\infty} \mathbf{A}^i \mathbf{b}$$

Equivalently, $(\mathbb{R}^N, g)$ is globally stable when

$$|\lambda_n| < 1, \quad \forall\, n,$$

where λ_n is the n-th eigenvalue of $\mathbf{A}$

Example. In the employment model, the dynamical system $(\mathbb{D}, g)$ is globally stable, in that it has a unique $\mathbf{p}^* \in \mathbb{D}$ satisfying

$$\mathbf{\Pi p}^* = \mathbf{p}^*$$

and, for any $\mathbf{p} \in \mathbb{D}$ we have $\mathbf{\Pi}^t \mathbf{p} \to \mathbf{p}^*$ as $t \to \infty$

The proof proceeds by showing that

- the state space $\mathbb{D}$ is a closed set
- the function $g(\mathbf{p}) = \mathbf{\Pi p}$ is a contraction mapping on $\mathbb{D}$

Here we'll just check that $\mathbb{D}$ is closed

Fact. $\mathbb{D}$ is a closed subset of $\mathbb{R}^2$

Proof: Recall that

$$\mathbb{D} = \left\{\mathbf{p} \in \mathbb{R}^2 \, : \, \mathbf{p} \geq \mathbf{0}, \; \mathbf{p}'\mathbf{1} = 1\right\}$$

Let $\{\mathbf{p}_n\}$ be a sequence in $\mathbb{D}$ converging to some $\mathbf{p} \in \mathbb{R}^2$

We claim that $\mathbf{p} \in \mathbb{D}$

To show this we need to verify that $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p}'\mathbf{1} = 1$

We showed in the last lecture that limits of nonnegative vectors are nonnegative

So here we'll just check that $\mathbf{p}'\mathbf{1} = 1$

To repeat, we have

$$\mathbf{p}_n \to \mathbf{p} \quad \text{and} \quad \mathbf{p}_n'\mathbf{1} = 1, \ \forall\, n \tag{1}$$

We claim that $\mathbf{p}'\mathbf{1} = 1$ must also hold

From an identical proof in lecture 25 we have

$$\mathbf{p}_n \to \mathbf{p} \quad \text{and} \quad \mathbf{p}_n'\mathbf{1} \leq 1, \ \forall\, n \quad \implies \quad \mathbf{p}'\mathbf{1} \leq 1 \tag{2}$$

A very similar argument gives

$$\mathbf{p}_n \to \mathbf{p} \quad \text{and} \quad \mathbf{p}_n'\mathbf{1} \geq 1, \ \forall\, n \quad \implies \quad \mathbf{p}'\mathbf{1} \geq 1 \tag{3}$$

By (1) both (2) and (3) are true, so $\mathbf{p}'\mathbf{1} \leq 1$ and $\mathbf{p}'\mathbf{1} \geq 1$

Hence $\mathbf{p}'\mathbf{1} = 1$ as claimed

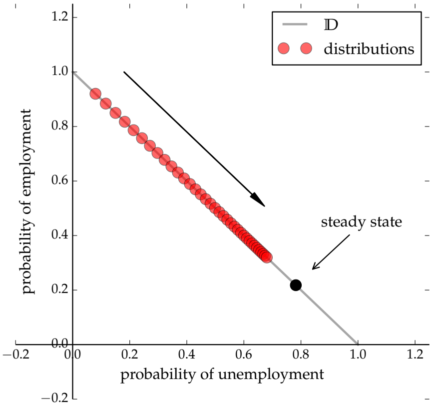


Figure : A trajectory and the unique steady state $\mathbf{p}^*$

Thanks for listening and good luck!