ECON2125/4021/8013

Lecture 21

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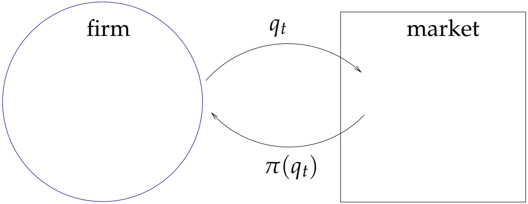
Application: A Planning Problem

A firm

- owns stock s_t of a natural resouce (e.g., oil)
- supplies q_t at time t and gets current profit $\pi(q_t)$



- stock next period is $s_{t+1} = s_t - q_t$

Suppose that $t = 0$, current stock is s_0

Given supply sequence $\{q_t\}_{t=0}^{\infty}$, net present value of profits flow is

$$\text{NPV} = \sum_{t=0}^{\infty} \beta^t \pi(q_t) \quad \text{where} \quad \beta := \frac{1}{1+r}$$

Assume the resource is nonrenewable, so

$$\text{sequence } \{q_t\} \text{ feasible} \quad \iff \quad \sum_{t=0}^{\infty} q_t \leq s_0$$

Suppose that

- s_t and q_t take integer values
- $\pi(q) = q^{\alpha}$ for some $\alpha \in (0, 1)$

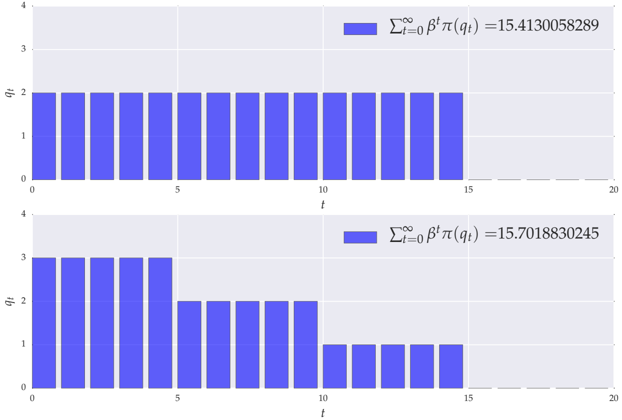


Figure : Present value of different $\{q_t\}$ sequences ($\alpha = 0.5$, $r = 0.05$)

Assume that the firm chooses $\{q_t\}$ to maximize NPV

Let $v^*(s)$ be the NPV corresponding to

- current stock s_0 equal to s
- an optimal supply sequence choice given $s = s_0$

$$v^*(s) = \sup \left\{ \sum_{t=0}^{\infty} \beta^t \pi(q_t) \; : \; \sum_{t=0}^{\infty} q_t \leq s \right\}$$

Thus $v^*(s)$ is the "market value of the firm with current stock s"

How to compute $v^*(s)$ for all $s \leq N =$: some max level of stock?

It turns out that v^* satisfies the equation

$$v^*(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v^*(s-q)\} \qquad (s = 0, \ldots, N)$$

Intuition: Max value attained if current q chosen to trade off

- current profits $\pi(q)$
- depletion of stock to $s - q$ weighted by future value

Proof: Omitted — see Bellman's principle of optimality

More intuition / examples of these kinds of recursions coming later

Remark: We're restricting q to be an integer for simplicity

Let $\mathbf{v} = (v(0), \ldots, v(N))$ be any vector in $\mathbb{R}^{N+1}$

Consider creating a new vector $\hat{\mathbf{v}} \in \mathbb{R}^{N+1}$ from $\mathbf{v}$ via

$$\hat{v}(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s-q)\} \qquad (s = 0, \ldots, N)$$

- $\hat{v}(0) = \max_{0 \leq q \leq 0}\{\pi(q) + \beta v(0-q)\} = \pi(0) + \beta v(0)$
- $\hat{v}(1) = \max_{0 \leq q \leq 1}\{\pi(q) + \beta v(1-q)\} = \cdots$
- $\cdots$

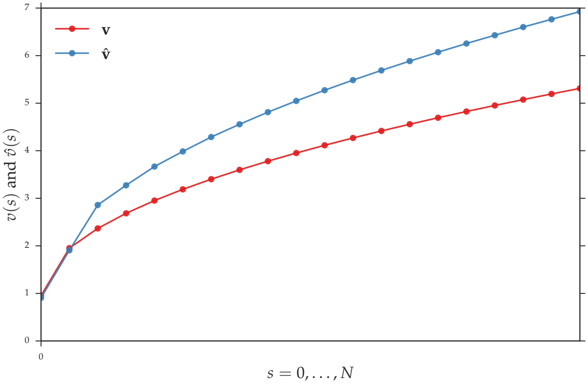


Figure : Creating $\hat{\mathbf{v}}$ from given $\mathbf{v}$

We've specified a rule that creates a new vector $\hat{\mathbf{v}}$ from any existing vector $\mathbf{v}$

We can think of this operation $\mathbf{v} \mapsto \hat{\mathbf{v}}$ as a mapping

Let T be the mapping defined in this way

That is, $\hat{\mathbf{v}} = T\mathbf{v}$ where

$$Tv(s) = \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s - q)\} \qquad (s = 0, 1, \ldots, N)$$

T is a well-defined mapping from $\mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$

Recall that

$$v^*(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v^*(s-q)\}$$

and that $T\colon \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ maps $\mathbf{v}$ to $\hat{\mathbf{v}}$ by

$$Tv(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v(s-q)\}$$

It follows that

$$Tv^*(s) = \max_{0 \le q \le s} \{\pi(q) + \beta v^*(s-q)\} = v^*(s)$$

That is, $T\mathbf{v}^* = \mathbf{v}^*$

Thus, solving for $\mathbf{v}^*$ is the same as finding a fixed point of T

Claim: T is a contraction on $\mathbb{R}^{N+1}$ with p-norm $\| \cdot \|_\infty$

Proof: Pick any $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^{N+1}$ and any s in $0, 1, \ldots, N$

By definition,

$$|Tv(s) - Tw(s)| =$$

$$\left| \max_{0 \leq q \leq s} \{\pi(q) + \beta v(s-q)\} - \max_{0 \leq q \leq s} \{\pi(q) + \beta w(s-q)\} \right|$$

Recall now the rule

$$|\sup_{x \in A} f(x) - \sup_{x \in A} g(x)| \leq \sup_{x \in A} |f(x) - g(x)|$$

Hence

$$
\begin{aligned}
|Tv(s) - Tw(s)| &\leq \max_{0 \leq q \leq s} |\pi(q) + \beta v(s-q) - (\pi(q) + \beta w(s-q))| \\
&= \beta \max_{0 \leq q \leq s} |v(s-q) - w(s-q)| \\
&\leq \beta \max_{0 \leq u \leq N} |v(u) - w(u)| \\
&= \beta \|\mathbf{v} - \mathbf{w}\|_\infty
\end{aligned}
$$

Since the last term is an upper bound on $|Tv(s) - Tw(s)|$, we have

$$
\|T\mathbf{v} - T\mathbf{w}\|_\infty \leq \beta \|\mathbf{v} - \mathbf{w}\|_\infty
$$

What we know so far

- T has a unique fixed point in $\mathbb{R}^{N+1}$
- that fixed point is $\mathbf{v}^*$, the object we want to compute
- If $\mathbf{v}$ is any point in $\mathbb{R}^{N+1}$, then $T^k \mathbf{v} \to \mathbf{v}^*$

So let's pick $\mathbf{v}$ and iterate with T

In practice we

1. Iterate until $\|T^k \mathbf{v} - T^{k+1} \mathbf{v}\|_\infty < \epsilon :=$ small error tolerance
2. Take the final $T^k \mathbf{v}$ as approximate solution for $\mathbf{v}^*$

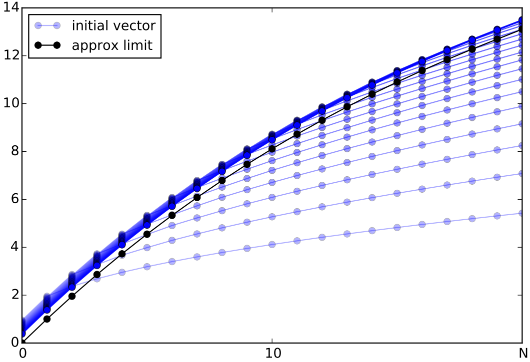


Figure : The sequence $\mathbf{v}, T\mathbf{v}, T^2\mathbf{v}, \ldots$ and limit

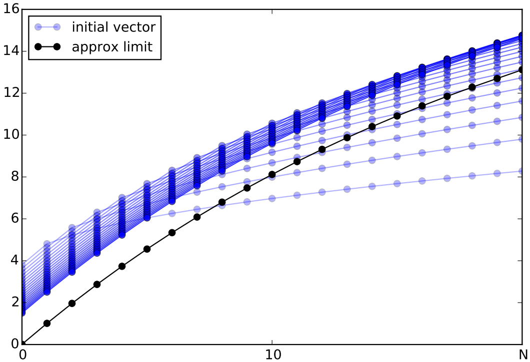


Figure : Iterates with alternative initial condition

Comparative Statics

Now we know how to compute a solution $\mathbf{v}^*$ for each set of parameters

Typical next step: look at the properties of the solution

Example. How is the value of the firm affected by r?

Intuitively, higher interest rate decreases net present value

Let's

- compute approximate $\mathbf{v}^*$ associated with different r
- see whether they do go down as r goes up

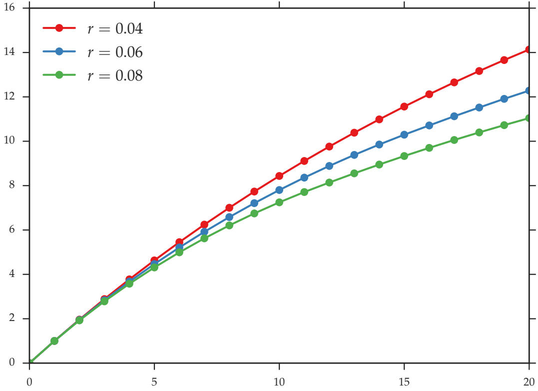


Figure : The vector $\mathbf{v}^*$ computed at different values of r

New Topic

DYNAMICS

Dynamics

Dynamics are essential to almost all areas of economics and finance

Why? Because the future matters for the present:

- Can't price an asset today without considering what it could be sold for tomorrow
- Can't analyze viability of a pension system without considering future time paths for income, savings, etc.
- Central banks can't choose interest rates without considering future inflation, unemployment and output

Introductory Example: Solow–Swan

We start with a simple example: Solow–Swan growth

1. Agents save some of their current income
2. Those savings are used to increase capital stock
3. Capital is combined with labour to produce output
4. Output is income (divided out as wages, rent on capital)
5. Return to step 1

What happens to output / capital / etc. over time?

In the model, output in each period is

$$Y_t = F(K_t, L_t) \qquad (t = 0, 1, 2, \ldots)$$

Here

- K_t = capital
- L_t = labor
- Y_t = output
- F is the aggregate production function

F assumed to be **homogeneous of degree one** (HD1), meaning

$$F(\lambda K, \lambda L) = \lambda F(K, L) \quad \text{for all } \lambda \geq 0$$

Examples.

Cobb-Douglas:

$$F(K, L) = AK^{\alpha}L^{1-\alpha}$$

CES:

$$F(K, L) = \gamma\{\alpha K^{\rho} + (1-\alpha)L^{\rho}\}^{1/\rho}$$

Closed economy:

$$\text{current domestic investment} \;=\; \text{aggregate domestic savings}$$

The savings rate is a positive constant s, so

$$\text{investment} \;=\; \text{savings} \;= sY_t = sF(K_t, L_t)$$

Depreciation means that 1 unit of capital today becomes $1 - \delta$ units next period

Thus, capital stock evolves according to

$$K_{t+1} = sF(K_t, L_t) + (1 - \delta)K_t$$

We simplify $K_{t+1} = sF(K_t, L_t) + (1-\delta)K_t$ as follows

Assume that $L_t =$ some constant L

Now set $k_t := K_t / L$ and use HD1 to get the per capita law of motion

$$k_{t+1} = s\frac{F(K_t, L)}{L} + (1-\delta)k_t$$

$$= sF(k_t, 1) + (1-\delta)k_t$$

Setting $f(k) := F(k, 1)$ to simplify notation, final expression is

$$k_{t+1} = sf(k_t) + (1-\delta)k_t$$

In summary, we can write

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sf(k) + (1 - \delta)k$$

This kind of equation is called a **difference equation**

In this case, scalar and nonlinear

Main question: what are the implied properties of $\{k_t\}$?

More generally, given

- difference equation $x_{t+1} = g(x_t)$
- initial condition x_0,

what are the properties of $\{x_t\}$?

45 Degree Diagrams

A method for tracing out dynamics graphically

Useful for analyzing one dimensional dynamic systems

Equally helpful for both linear and nonlinear systems

Let's look at some examples, starting with the difference equation

$$x_{t+1} = g(x_t) \quad \text{when} \quad g(x) = 2 + 0.5x$$

We want to be able to take any x_0 and map out the sequence

$$x_0, \quad x_1 = g(x_0), \quad x_2 = g(x_1), \quad \dots$$

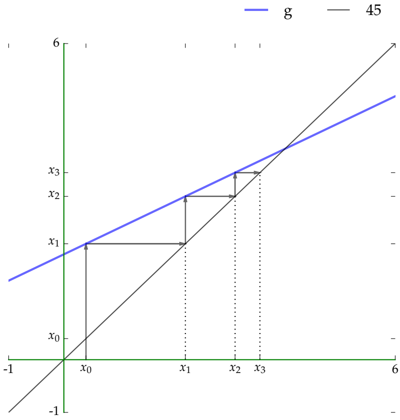


Figure : $g(x) = 2 + 0.5x$ with $x_0 = 0.4$

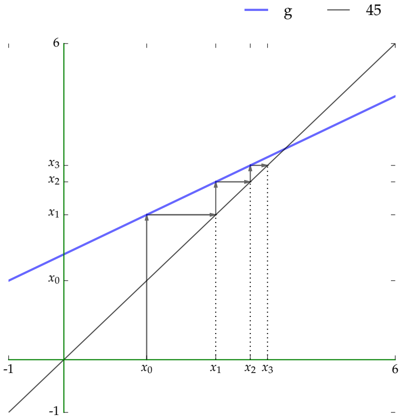


Figure : $g(x) = 2 + 0.5x$ with $x_0 = 1.5$

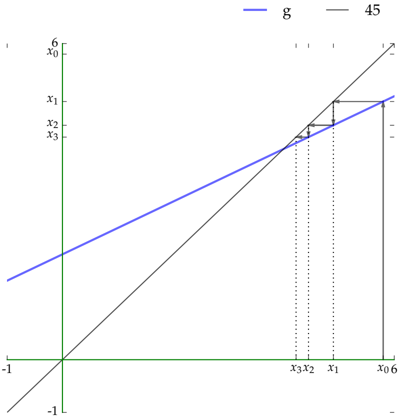


Figure : $g(x) = 2 + 0.5x$ with $x_0 = 5.8$

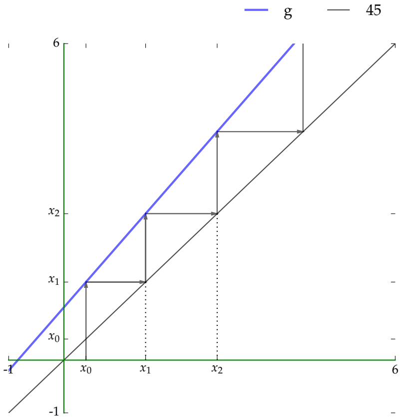


Figure : $g(x) = 1 + 1.2x$ with $x_0 = 0.4$

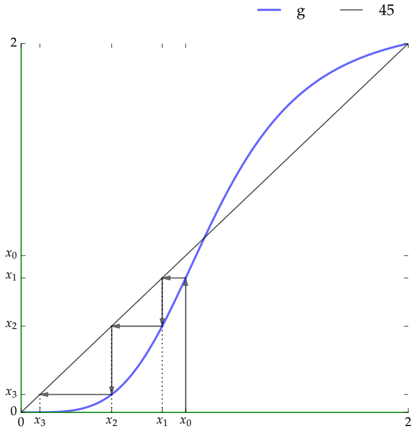


Figure : $g(x) = 2.125/(1 + x^{-4})$ with $x_0 = 0.85$

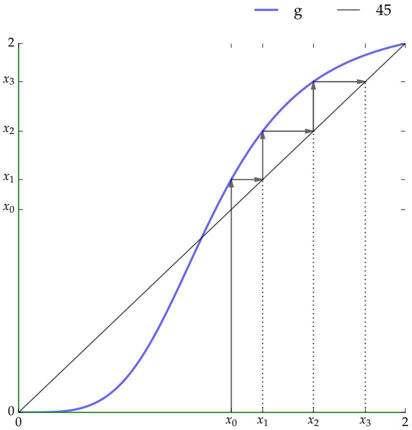
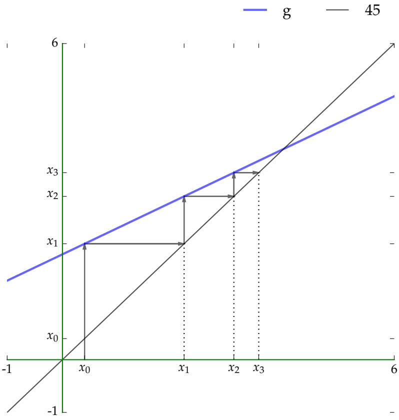


Figure : $g(x) = 2.125/(1 + x^{-4})$ with $x_0 = 1.1$

Let's compare

- 45 degree diagrams
- corresponding time series plots



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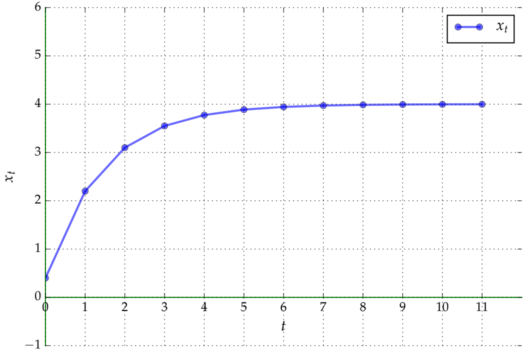

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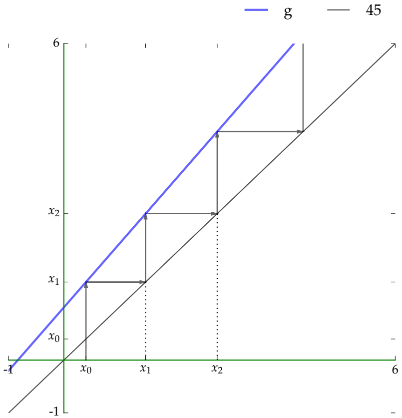
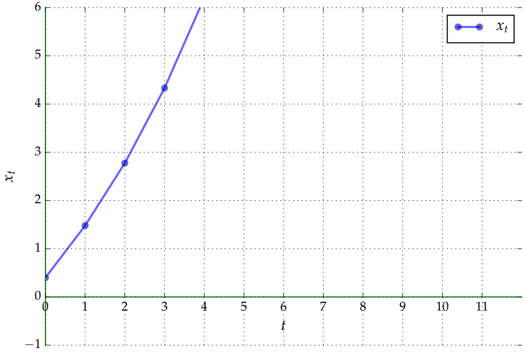


Figure : $g(x) = 1 + 1.2x$ with $x_0 = 0.4$



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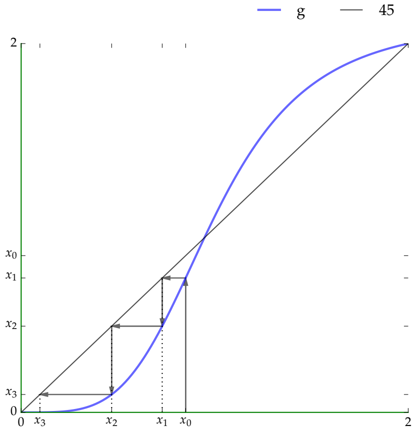
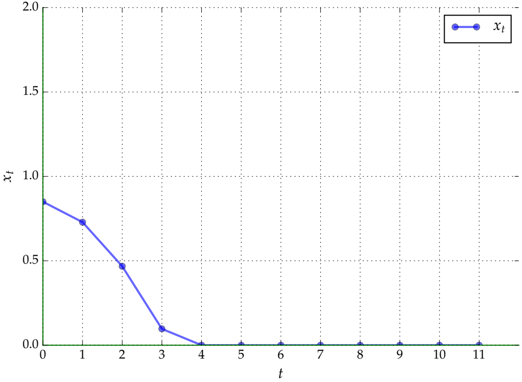


Figure : $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 0.85$



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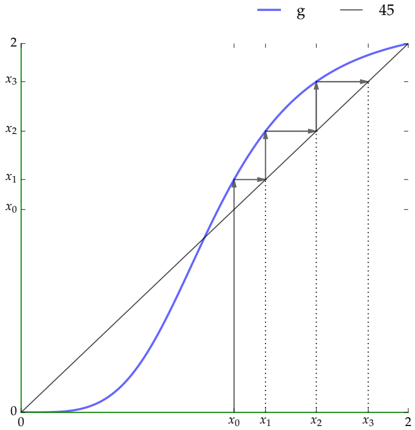


Figure : $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 1.1$

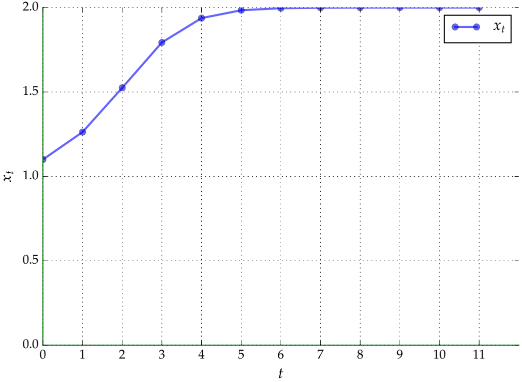


Figure : $g(x) = 2.125/(1 + x^{-4})$ and $g(0) = 0$ with $x_0 = 1.1$

Back to Solow-Swan

Let's return to the model

$$k_{t+1} = g(k_t) \quad \text{where} \quad g(k) := sf(k) + (1-\delta)k$$

Let's assume that

- $f(k) = Ak^{\alpha}$ where $A = 1$ and $\alpha = 0.6$
- $s = 0.3$ and $\delta = 0.1$

The dynamics can be seen graphically

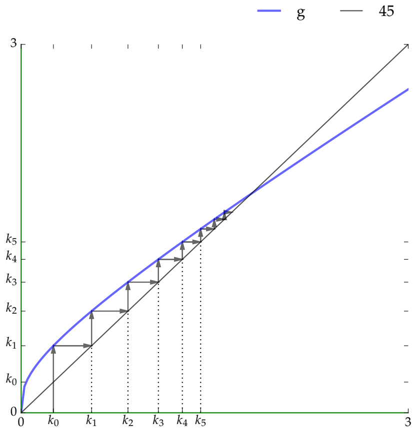
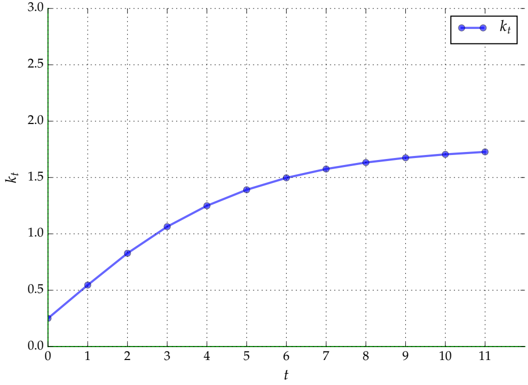


Figure : Solow-Swan dynamics, low initial capital



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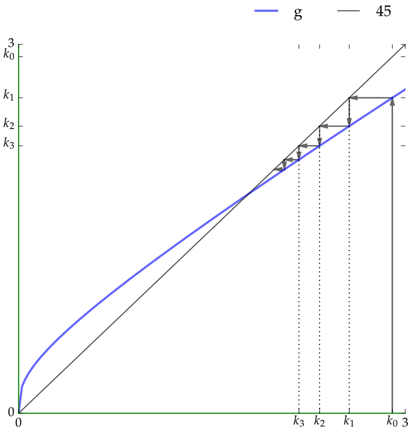

Figure : Solow-Swan dynamics, high initial capital

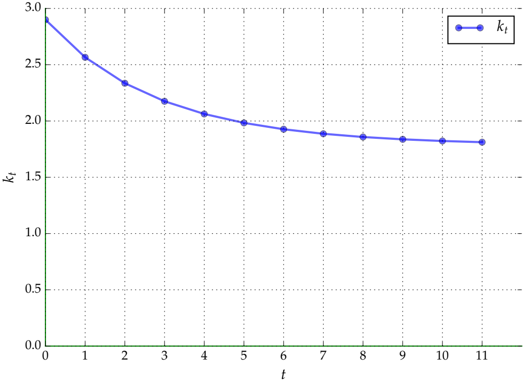

Figure : Solow-Swan dynamics, high initial capital

Graphical analysis of the model suggests that

- k_t increases over time if k_0 is small
- k_t decreases over time if k_0 is large
- k_t converges to the same point regardless of k_0

To go further with our analysis we need some definitions and results...