ECON2125/8013

Lecture 2

John Stachurski

Semester 1, 2015

Today's tasks

- Some comments on computing

Review / introduce some basic tools for problem solving

- Univariate optimization
- Working with multivariate functions
- Multivariate optimization

Comments on Computing

The way we do mathematics is changing

Example. In 1944, Hans Bethe solved following problem by hand

- will detonating an atom bomb ignite the atmosphere and thereby destroy life on earth?

These days we rarely calculate with actual numbers

Almost all calculations are done on computers

Example. Numerical integration

$$\frac{1}{\sqrt{2\pi}} \int_{-2}^{2} \exp\left\{-\frac{x^2}{2}\right\} dx$$

```
In [1]: from scipy.stats import norm
In [2]: from scipy.integrate import quad
In [3]: phi = norm()
In [4]: value, error = quad(phi.pdf, -2, 2)
In [5]: value
Out[5]: 0.9544997361036417
```

Example. Numerical optimization

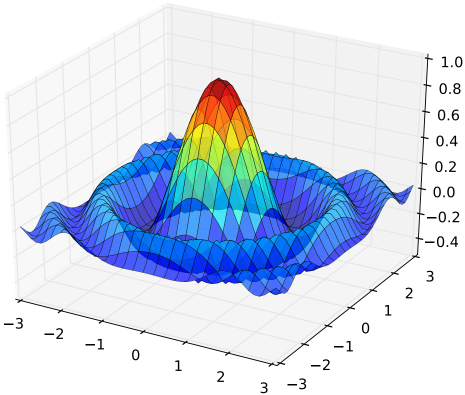
$$f(x) := -\exp\left\{-\frac{(x-5.0)^4}{1.5}\right\}$$

```
In [1]: from scipy.optimize import fminbound
In [2]: import numpy as np
In [3]: def f(x): return -np.exp(-(x - 5.0)**4 / 1.5)
In [4]: fminbound(f, -10, 10)  # Find approx solution
Out[4]: 4.999941901210506
```

Example. Visualization

What does this function look like?

$$f(x,y) = \frac{\cos(x^2 + y^2)}{1 + x^2 + y^2}$$


1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−3
−2
−1
0
1
2
3
3
2
1
0
−1
−2
−3

Example. Symbolic calculations

Let's differentiate $f(x) = (1 + 2x)^5$

Forgotten how? No problems:

```
In [1]: import sympy as sp
In [2]: x = sp.Symbol('x')
In [3]: fx = (1 + 2 * x)**5
In [4]: fx.diff(x)
Out[4]: 10*(2*x + 1)**4
```

So if computers can do our maths for us, why learn maths?

The difficulty is

- giving them the right inputs and instructions
- interpreting what comes out

The skills we need are

- Understanding of fundamental concepts
- Sound deductive reasoning

These are the focus of the course

Computer Code in the Lectures

While computation is not a formal part of the course...

Throughout the course I'll inject little bits of code into the course

All the code will be written in the Python programming language

- This is meant to illustrate the kinds of things we can do
- It is not assessable

You might find value in actually running the code shown in lectures

(If not for the course then more generally)

Python and all its scientific code libraries are free to install

If you want to do so please refer to

`http://quant-econ.net/py/index.html`

In particular,

`http://quant-econ.net/py/getting_started.html`

Univariate Optimization – A Review

Let $f\colon [a,b] \to \mathbb{R}$ be a differentiable (smooth) function

Here:

- $[a,b]$ is all x with $a \leq x \leq b$
- $\mathbb{R}$ is "all numbers"
- f takes $x \in [a,b]$ and returns number $f(x)$
- derivative $f'(x)$ exists for all x with $a < x < b$

A point $x^* \in [a, b]$ is called a

- **maximizer** of f on $[a, b]$ if $f(x^*) \geq f(x)$ for all $x \in [a, b]$
- **minimizer** of f on $[a, b]$ if $f(x^*) \leq f(x)$ for all $x \in [a, b]$

Example. Let

- $f(x) = -(x-4)^2 + 10$
- $a = 2$ and $b = 8$

Then

- $x^* = 4$ is a maximizer of f on $[2, 8]$
- $x^{**} = 8$ is a minimizer of f on $[2, 8]$

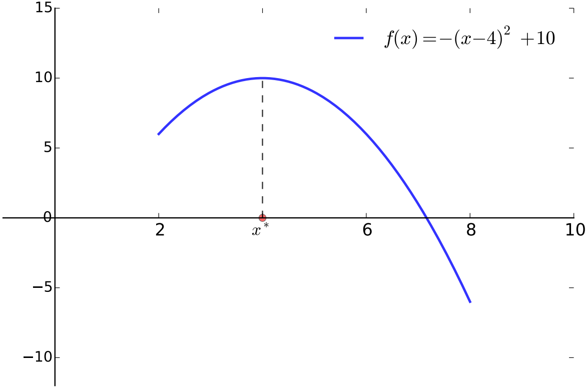


Figure : Maximizer on $[a,b] = [2,8]$ is $x^* = 4$

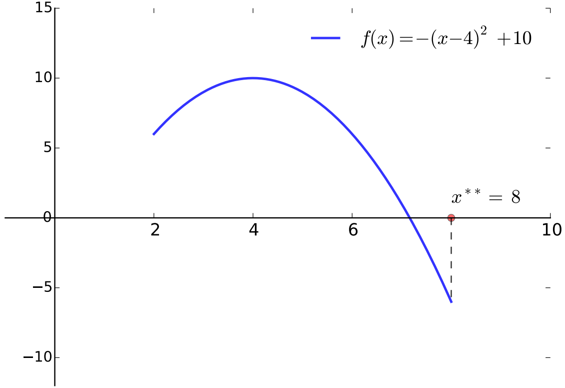


Figure : Minimizer on $[a, b] = [2, 8]$ is $x^{**} = 8$

The set of maximizers/minimizers can be

- empty
- a singleton
- infinite

Example. $f\colon [0,1] \to \mathbb{R}$ defined by $f(x) = 1$ has infinitely many maximizers and minimizers on $[0,1]$

Example. The following function has no maximizers on $[0,2]$

$$f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 1/2 & \text{otherwise} \end{cases}$$

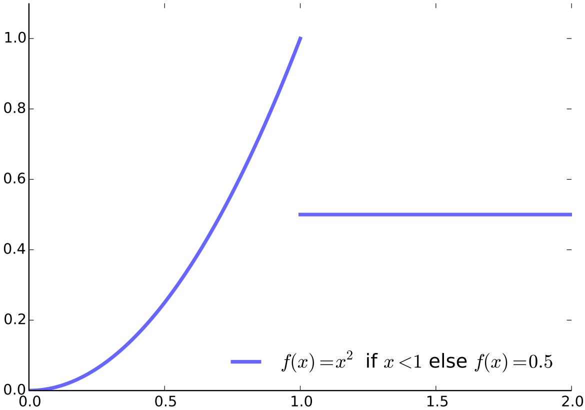


Figure : No maximizer on $[0, 2]$

Point x is called **interior** to $[a, b]$ if $a < x < b$

The set of all interior points is written (a, b)

A point $x^* \in [a, b]$ is called an

- **interior maximizer** if both a maximizer and interior
- **interior minimizer** if both a minimizer and interior

Finding Optima

A **stationary point** of f on $[a, b]$ is an interior point x with $f'(x) = 0$

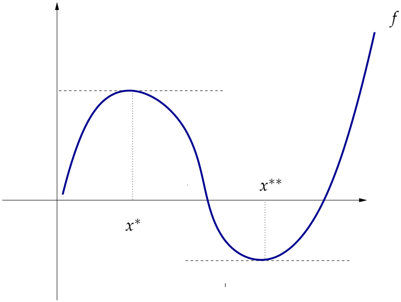


Figure : Both x^* and x^{**} are stationary

Fact. If f is differentiable and x^* is either an interior minimizer or an interior maximizer of f on $[a, b]$, then x^* is stationary

Sketch of proof, for maximizers:

$$f'(x^*) = \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} \qquad \text{(by def.)}$$

$$\therefore \quad f(x^* + h) \approx f(x^*) + f'(x^*)h \qquad \text{for small } h$$

If $f'(x^*) \neq 0$ then exists small h such that $f(x^* + h) > f(x^*)$

Hence interior maximizers must be stationary — otherwise can do better

$$\therefore \quad \text{any interior maximizer stationary}$$

$$\therefore \quad \text{set of interior maximizers} \subset \text{set of stationary points}$$

$$\therefore \quad \text{maximizers} \subset \text{stationary points} \cup \{a\} \cup \{b\}$$

Usage:

- Locate stationary points
- Evaluate $y = f(x)$ for each stationary x and for a, b
- Pick point giving largest y value

Minimization: Same idea

Example

Let's solve

$$\max_{-2 \leq x \leq 5} f(x) \quad \text{where} \quad f(x) = x^3 - 6x^2 + 4x + 8$$

Steps

- Differentiate to get $f'(x) = 3x^2 - 12x + 4$
- Solve $3x^2 - 12x + 4 = 0$ to get stationary x
- Discard any stationary points outside $[-2, 5]$
- Eval f at remaining points plus end points -2 and 5
- Pick point giving largest value

```
from sympy import *
x = Symbol('x')
points = [-2, 5]
f = x**3 - 6*x**2 + 4*x + 8
fp = diff(f, x)
spoints = solve(fp, x)
points.extend(spoints)
v = [f.subs(x, c).evalf() for c in points]
msg = "Maximizer = "
print msg + str(points[v.index(max(v))])
```

Prints: `Maximizer = 2 - 2*sqrt(6)/3`, which is ≈ 0.367

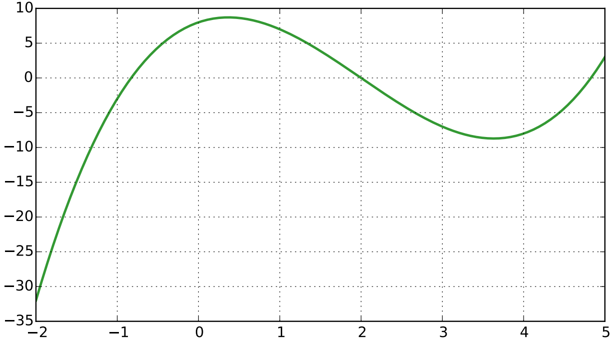
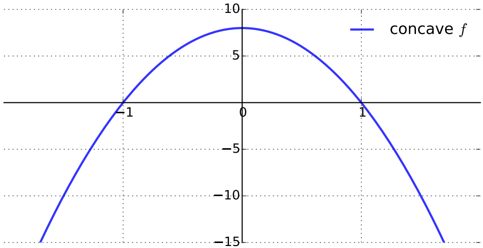


Figure : Maximizer is ≈ 0.367

Shape Conditions and Sufficiency

When is $f'(x^*) = 0$ sufficient for x^* to be a maximizer?

One answer: When f is concave



(Full definition deferred — sufficient conditions below)

Sufficient conditions for **concavity** in one dimension

Let $f\colon [a,b] \to \mathbb{R}$

Facts

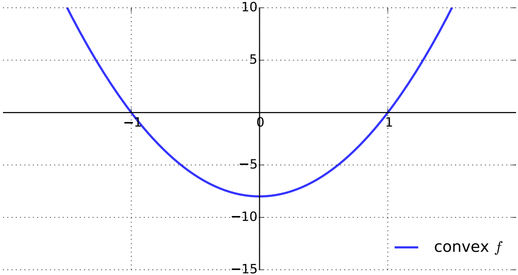
- If $f''(x) \leq 0$ for all $x \in (a,b)$ then f is concave on (a,b)
- If $f''(x) < 0$ for all $x \in (a,b)$ then f is strictly concave on (a,b)

Examples.

- $f(x) = a + bx$ is concave on $\mathbb{R}$ but not strictly
- $f(x) = \log(x)$ is strictly concave on $(0,\infty)$

When is $f'(x^*) = 0$ sufficient for x^* to be a minimizer?

One answer: When f is convex



(Full definition deferred — sufficient conditions below)

Sufficient conditions for **convexity** in one dimension

Let $f\colon [a,b] \to \mathbb{R}$

Facts

- If $f''(x) \geq 0$ for all $x \in (a,b)$ then f is convex on (a,b)
- If $f''(x) > 0$ for all $x \in (a,b)$ then f is strictly convex on (a,b)

Examples.

- $f(x) = a + bx$ is convex on $\mathbb{R}$ but not strictly
- $f(x) = x^2$ is strictly convex on $\mathbb{R}$

Facts for maximizers

- If $f\colon [a,b] \to \mathbb{R}$ is concave and $x^* \in (a,b)$ is stationary then x^* is a maximizer
- If, in addition, f is strictly concave, then x^* is the unique maximizer

Facts for minimizers

- If $f\colon [a,b] \to \mathbb{R}$ is convex and $x^* \in (a,b)$ is stationary then x^* is a minimizer
- If, in addition, f is strictly convex, then x^* is the unique minimizer

Example

A price taking firm faces output price $p > 0$, input price $w > 0$

Maximize profits with respect to input ℓ

$$\max_{\ell \geq 0} \pi(\ell) = pf(\ell) - w\ell$$

- $f(\ell) = \ell^\alpha$ with $0 < \alpha < 1$

Evidently $\pi'(\ell) = \alpha p \ell^{\alpha - 1} - w$ so unique stationary point is

$$\ell^* := (\alpha p / w)^{1/(1-\alpha)}$$

Moreover $\pi''(\ell) = \alpha(\alpha - 1) p \ell^{\alpha - 2} < 0$ for all ℓ so ℓ^* is unique maximizer

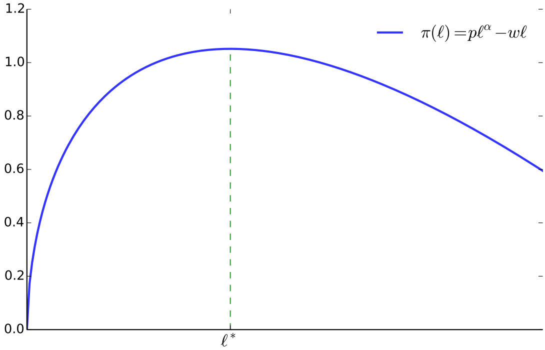


Figure : Profit maximization with $p = 2$, $w = 1$, $\alpha = 0.6$

Functions of Two Variables

Let's have a look at some functions of two variables

- How to visualize them
- Slope, contours, etc.

Example. Consider production function

$$f(k, \ell) = k^{\alpha} \ell^{\beta}$$

$$0 \leq \alpha, \beta \qquad \alpha + \beta < 1$$

Let's graph it in two dimensions

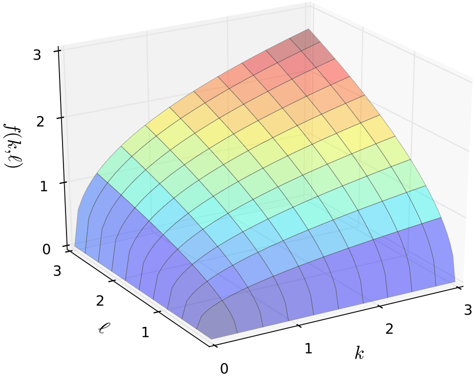


Figure : Production function with $\alpha = 0.4$, $\beta = 0.5$

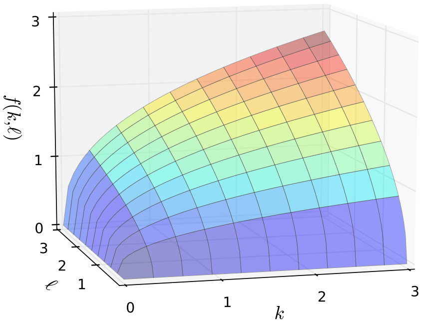


Figure : Production function with $\alpha = 0.4$, $\beta = 0.5$

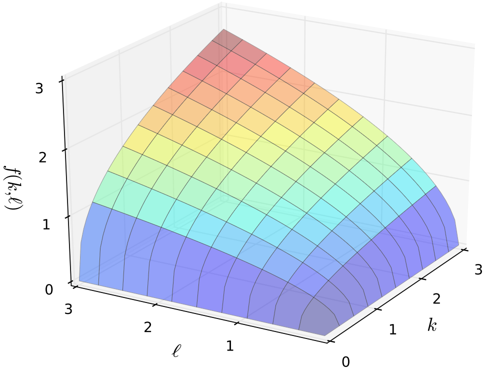


Figure : Production function with $\alpha = 0.4$, $\beta = 0.5$

Like many 3D plots it's hard to get a good understanding

Let's try again with contours plus heat map

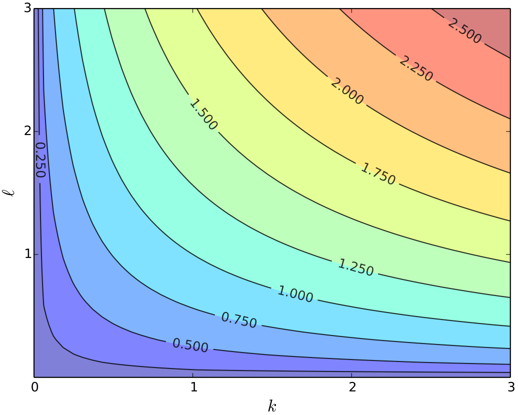


Figure : Production function with $\alpha = 0.4$, $\beta = 0.5$

(In this context the contour lines are called **isoquants**)

Can you see how $\alpha < \beta$ shows up in the slope of the contours?

We can drop the colours to see the numbers more clearly

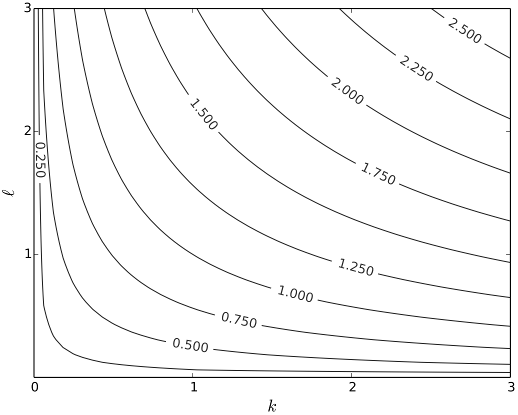


Figure : Production function with $\alpha = 0.4$, $\beta = 0.5$

Example. Let $u(x_1, x_2)$ be "utility" gained from x_1 units of good 1 and x_2 units of good 2

We take

$$u(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$$

where

- α and β are parameters
- We assume $\alpha, \beta > 0$
- The log functions mean "diminishing returns" in each good

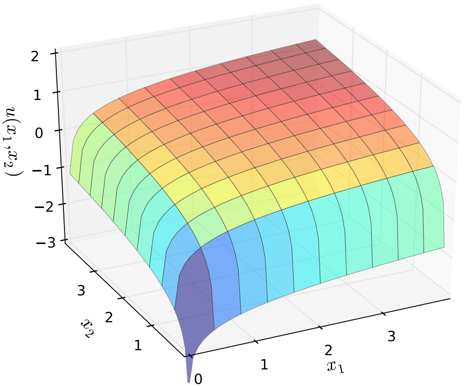


Figure : Log utility with $\alpha = 0.4$, $\beta = 0.5$

Let's look at the contour lines

For utility functions, contour lines called **indifference curves**

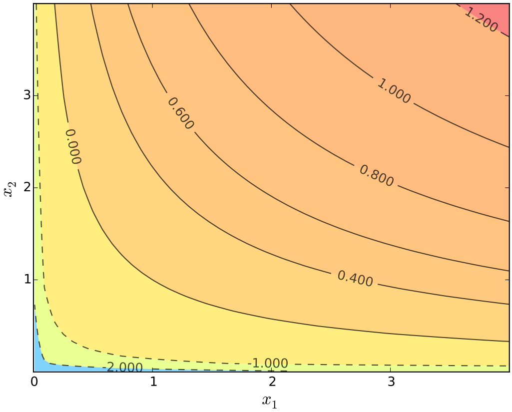


Figure : Log utility with $\alpha = 0.4$, $\beta = 0.5$

Another example: Quasi-linear utility function, two goods

$$u(x_1, x_2) = x_1 + \log(x_2)$$

- Called quasi-linear because linear in good 1

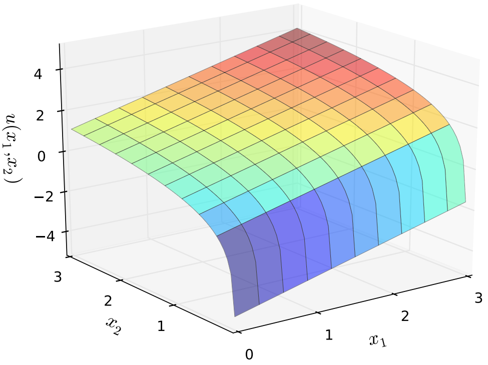


Figure : Quasi-linear utility

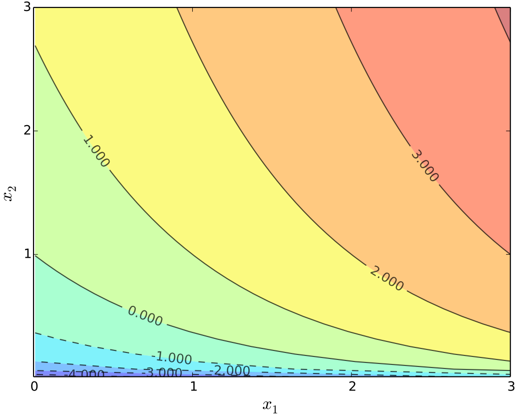


Figure : Quasi-linear utility

Another example: Quadratic utility, two goods

$$u(x_1, x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2$$

Here

- b_1 is a "satiation" or "bliss" point for x_1
- b_2 is a "satiation" or "bliss" point for x_2

Dissatisfaction increases with deviations from the bliss points

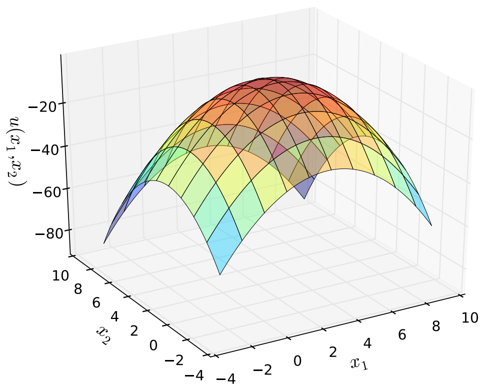


Figure : Quadratic utility with $b_1 = 3$ and $b_2 = 2$

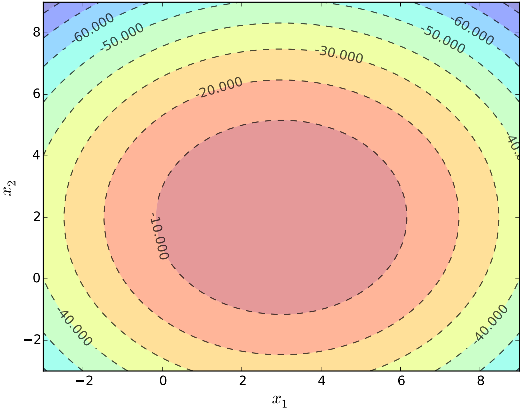


Figure : Quadratic utility with $b_1 = 3$ and $b_2 = 2$

Bivariate Optimization

Consider $f\colon I \to \mathbb{R}$ where $I \subset \mathbb{R}^2$

The set $\mathbb{R}^2$ is all (x_1, x_2) pairs

A point $(x_1^*, x_2^*) \in I$ is called a **maximizer** of f on I if

$$f(x_1^*, x_2^*) \geq f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in I$$

A point $(x_1^*, x_2^*) \in I$ is called a **minimizer** of f on I if

$$f(x_1^*, x_2^*) \leq f(x_1, x_2) \quad \text{for all} \quad (x_1, x_2) \in I$$

When they exist, the partial derivatives at $(x_1, x_2) \in I$ are

$$f_1(x_1, x_2) := \frac{\partial}{\partial x_1} f(x_1, x_2)$$

$$f_2(x_1, x_2) := \frac{\partial}{\partial x_2} f(x_1, x_2)$$

Example. When $f(k, \ell) = k^\alpha \ell^\beta$,

$$f_1(k, \ell) = \frac{\partial}{\partial k} f(k, \ell) = \frac{\partial}{\partial k} k^\alpha \ell^\beta = \alpha k^{\alpha-1} \ell^\beta$$

An interior point $(x_1, x_2) \in I$ is called **stationary** for f if

$$f_1(x_1, x_2) = f_2(x_1, x_2) = 0$$

Fact. Let $f\colon I \to \mathbb{R}$ be a continuously differentiable function. If (x_1^*, x_2^*) is either

- an interior maximizer of f on I, or
- an interior minimizer of f on I,

then (x_1^*, x_2^*) is a stationary point of f

Usage, for maximization:

1. Compute partials
2. Set partials to zero to find $S :=$ all stationary points
3. Evaluate candidates in S and boundary of I
4. Select point (x_1^*, x_2^*) yielding highest value

Example. Testing on an obvious example:

$$\min f(x_1, x_2) = x_1^2 + 4x_2^2 \quad \text{s.t.} \quad x_1 + x_2 \leq 1$$

Setting

$$f_1(x_1, x_2) = 2x_1 = 0 \quad \text{and} \quad f_2(x_1, x_2) = 8x_2 = 0$$

gives the unique stationary point $(0, 0)$, at which $f(0, 0) = 0$

On the boundary we have $x_1 + x_2 = 1$, so

$$f(x_1, x_2) = f(x_1, 1 - x_1) = x_1^2 + 4(1 - x_1)^2$$

Ex. Show right hand side > 0 for any x_1

Hence minimizer is $(x_1^*, x_2^*) = (0, 0)$

Nasty secrets

Solving for (x_1, x_2) such that $f_1(x_1, x_2) = 0$ and $f_2(x_1, x_2) = 0$ can be hard

- System of nonlinear equations
- Might have no analytical solution
- Set of solutions can be a continuum

Example. (Don't) try to find all stationary points of

$$f(x_1, x_2) = \frac{\cos(x_1^2 + x_2^2) + x_1^2 + x_1}{2 + \exp(-x_1^2) + \sin^2(x_2)}$$

Also:

- Boundary is often a continuum, not just two points
- Things get even harder in higher dimensions

On the other hand:

- Most classroom examples are chosen to avoid these problems
- Life is still pretty easy if we have concavity / convexity
- Clever tricks have been found for certain kinds of problems

Second Order Partials

Let $f\colon I \to \mathbb{R}$ and, when they exist, let

$$f_{11}(x_1, x_2) := \frac{\partial^2}{\partial x_1^2} f(x_1, x_2)$$

$$f_{12}(x_1, x_2) := \frac{\partial^2}{\partial x_1 \partial x_2} f(x_1, x_2)$$

$$f_{21}(x_1, x_2) := \frac{\partial^2}{\partial x_2 \partial x_1} f(x_1, x_2)$$

$$f_{22}(x_1, x_2) := \frac{\partial^2}{\partial x_2^2} f(x_1, x_2)$$

Example. If $\pi(k, \ell) := pk^{\alpha}\ell^{\beta} - w\ell - rk$ then

$$\pi_{11}(k, \ell) = p\alpha(\alpha - 1)k^{\alpha-2}\ell^{\beta}$$

Fact. If $f \colon I \to \mathbb{R}$ is twice continuously differentiable at (x_1, x_2), then

$$f_{12}(x_1, x_2) = f_{21}(x_1, x_2)$$

Ex. Confirm that

$$\pi_{12}(k, \ell) = \pi_{21}(k, \ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$$

Shape Conditions

Let I be an "open" set (only interior points – formalities later)

Let $f\colon I \to \mathbb{R}$ be twice continuously differentiable

The function f is strictly concave on I if, for any $(x_1, x_2) \in I$,

1. $f_{11}(x_1, x_2) < 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

The function f is strictly convex on I if, for any $(x_1, x_2) \in I$,

1. $f_{11}(x_1, x_2) > 0$
2. $f_{11}(x_1, x_2)\, f_{22}(x_1, x_2) > f_{12}(x_1, x_2)^2$

When is stationarity sufficient?

Fact. If f is differentiable and strictly concave on I, then any stationary point of f is also a unique maximizer of f on I

Fact. If f is differentiable and strictly convex on I, then any stationary point of f is also a unique minimizer of f on I

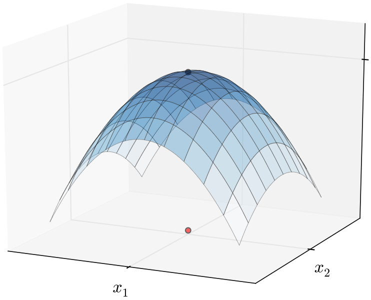


Figure : Maximizer of a concave function

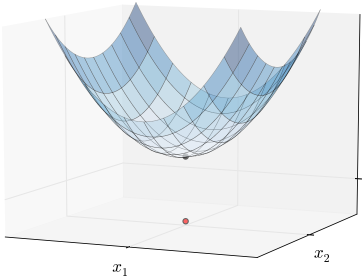


Figure : Minimizer of a convex function

Example. Quadratic utility, unconstrained

$$\max_{x_1,x_2} u(x_1,x_2) = -(x_1 - b_1)^2 - (x_2 - b_2)^2$$

Intuitively the solution is $x_1^* = b_1$ and $x_2^* = b_2$

Analysis above leads to the same conclusion

First let's check first order conditions

$$\frac{\partial}{\partial x_1} u(x_1,x_2) = -2(x_1 - b_1) = 0 \quad \implies \quad x_1 = b_1$$

$$\frac{\partial}{\partial x_2} u(x_1,x_2) = -2(x_2 - b_2) = 0 \quad \implies \quad x_2 = b_2$$

How about strict concavity?

Sufficient condition is

1. $u_{11}(x_1, x_2) < 0$
2. $u_{11}(x_1, x_2)u_{22}(x_1, x_2) > u_{12}(x_1, x_2)^2$

Here

1. $u_{11}(x_1, x_2) = -2$
2. $u_{11}(x_1, x_2)u_{22}(x_1, x_2) = 4 > 0 = u_{12}(x_1, x_2)^2$

Example. Profit maximization with two inputs

$$\max_{k,\ell} \pi(k,\ell) := pk^{\alpha}\ell^{\beta} - w\ell - rk$$

where α, β, p, w are all > 0 and $\alpha + \beta < 1$

Derivatives:

- $\pi_1(k,\ell) = p\alpha k^{\alpha-1}\ell^{\beta} - r$
- $\pi_2(k,\ell) = p\beta k^{\alpha}\ell^{\beta-1} - w$
- $\pi_{11}(k,\ell) = p\alpha(\alpha-1)k^{\alpha-2}\ell^{\beta}$
- $\pi_{22}(k,\ell) = p\beta(\beta-1)k^{\alpha}\ell^{\beta-2}$
- $\pi_{12}(k,\ell) = p\alpha\beta k^{\alpha-1}\ell^{\beta-1}$

First order conditions: Set

$$\pi_1(k, \ell) = 0$$

$$\pi_2(k, \ell) = 0$$

and solve simultaneously for k, ℓ to get

$$k^* = \left[p(\alpha/r)^{1-\beta}(\beta/w)^{\beta}\right]^{1/(1-\alpha-\beta)}$$

$$\ell^* = \left[p(\beta/w)^{1-\alpha}(\alpha/r)^{\alpha}\right]^{1/(1-\alpha-\beta)}$$

Ex. Verify

Now we check second order conditions, hoping for strict concavity

What we need: For any $k, \ell > 0$,

$$\pi_{11}(k, \ell) < 0 \quad \text{and} \quad \pi_{11}(k, \ell)\, \pi_{22}(k, \ell) > \pi_{12}(k, \ell)^2$$

Ex. Show both inequalities satisfied when $\alpha + \beta < 1$

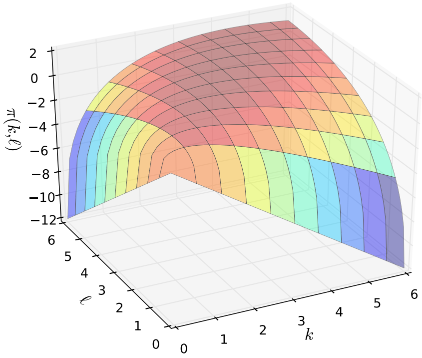


Figure : Profit function when $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$

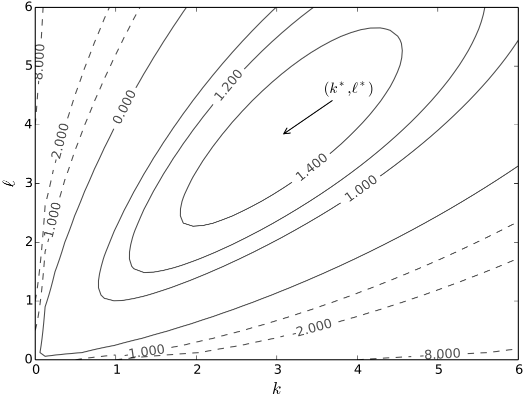


Figure : Optimal choice, $p = 5$, $r = w = 2$, $\alpha = 0.4$, $\beta = 0.5$