ECON2125/4021/8013

Lecture 18

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Fundamentals of Optimization

In elementary econ / finance courses we get well behaved, prepackaged problems

Usually they

- have a solution
- the solution is unique and not hard to find

We discussed such problems in the first few lectures

However, when we tackle new proplems such properties aren't guaranteed

We need some idea of how to check these things

Suprema and Infima

Consider the problem of finding the "maximum" or "minimum" of a function

A first issue is that such values might not be well defined

This leads us to start with "suprema" and "infima"

- Always well defined
- Agree with max and min when the latter exist

Let $A \subset \mathbb{R}$

A number $u \in \mathbb{R}$ is called an **upper bound** of A if

$$a \leq u \quad \text{for all} \quad a \in A$$

Example. If $A = (0,1)$ then 10 is an upper bound of A

$\because$ Every element of $(0,1)$ is ≤ 10

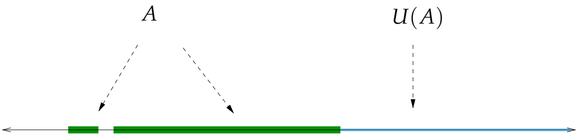
Example. If $A = (0,1)$ then 1 is an upper bound of A

$\because$ Every element of $(0,1)$ is ≤ 1

Example. If $A = (0,1)$ then 0.5 is <u>not</u> an upper bound of A

$\because$ $0.6 \in (0,1)$ and $0.5 < 0.6$

Let $U(A) :=$ set of all upper bounds of A



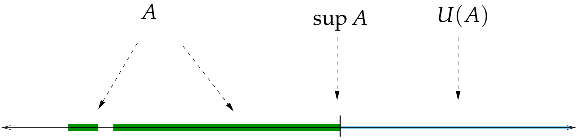
Examples.

- If $A = [0, 1]$, then $U(A) = [1, \infty)$
- If $A = (0, 1)$, then $U(A) = [1, \infty)$
- If $A = (0, 1) \cup (2, 3)$, then $U(A) = [3, \infty)$
- If $A = \mathbb{N}$, then $U(A) = \emptyset$

If s is a number satisfying

$$s \in U(A) \qquad \text{and} \qquad s \leq u, \ \forall\, u \in U(A)$$

then s is called the **supremum** of A and we write $s = \sup A$



Also called the **least upper bound** of A

Example. If $A = (0, 1]$, then $U(A) = [1, \infty)$, so $\sup A = 1$

Example. If $A = (0, 1)$, then $U(A) = [1, \infty)$, so $\sup A = 1$

A set $A \subset \mathbb{R}$ is called **bounded above** if $U(A)$ is not empty

Fact. If A is nonempty and bounded above then A has a supremum in $\mathbb{R}$

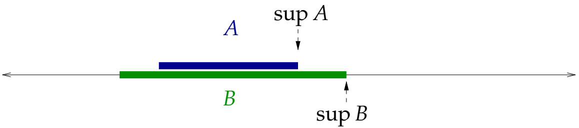
- Equivalent to the fact that all Cauchy sequences converge
- Same principle: $\mathbb{R}$ has no "gaps" or "holes"

What if A is not bounded above, so that $U(A) = \emptyset$?

We follow the convention that $\sup A := \infty$ in this case

Now the supremum of a nonempty subset of $\mathbb{R}$ **always** exists

Fact. If $A \subset B$, then $\sup A \leq \sup B$



Proof: Let $A \subset B$

If $\sup B = \infty$ then the claim is trivial so suppose $\bar{b} = \sup B < \infty$

By definition, $\bar{b} \in U(B)$, so $b \leq \bar{b}$ for all $b \in B$

Since each $a \in A$ is also in B, we then have $a \leq \bar{b}$ for all $a \in A$

It follows that $\bar{b} \in U(A)$

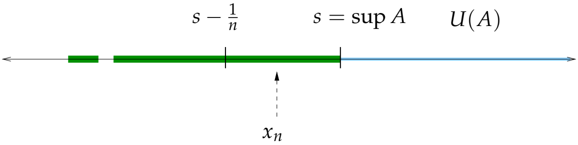
Hence $\sup A \leq \bar{b}$

Let A be any set bounded from above and let $s := \sup A$

Fact. There exists a sequence $\{x_n\}$ in A with $x_n \to s$

Proof: Note that

$$\forall\, n \in \mathbb{N}, \;\; \exists\, x_n \in A \;\; \text{s.t.} \;\; x_n > s - \frac{1}{n}$$



(Otherwise s is not a sup, because $s - \frac{1}{n}$ is a smaller upper bound)

The sequence $\{x_n\}$ lies in A and converges to s

A **lower bound** of $A \subset \mathbb{R}$ is any $\ell \in \mathbb{R}$ with $\ell \leq a$ for all $a \in A$

If $i \in \mathbb{R}$ is an lower bound for A with $i \geq \ell$ for every lower bound ℓ of A, then i is called the **infimum** of A

Write $i = \inf A$

Examples.

- If $A = [0,1]$, then $\inf A = 0$
- If $A = (0,1)$, then $\inf A = 0$

Fact. Every nonempty subset of $\mathbb{R}$ bounded from below has an infimum

If A is unbounded below then we set $\inf A = -\infty$

Maxima and Minima of Sets

In optimization we're mainly interested in maximizing / minimizing functions

If we maximize a function, say, then the problem looks like

$$\max_{\mathbf{x} \in A} f(\mathbf{x})$$

As we'll see, the problem is the same as finding the largest number in the range of f

That is, the largest number in the set

$$f(A) := \{f(\mathbf{x}) : \mathbf{x} \in A\}$$

So let's start by thinking about the largest value in a set

We call a^* the **maximum** of $A \subset \mathbb{R}$ and write $a^* = \max A$ if

$$a^* \in A \qquad \text{and} \qquad a \leq a^* \text{ for all } a \in A$$

- Example. If $A = [0, 1]$ then $\max A = 1$

We call a^* the **minimum** of $A \subset \mathbb{R}$ and write $a^* = \min A$ if

$$a^* \in A \qquad \text{and} \qquad a^* \leq a \text{ for all } a \in A$$

- Example. If $A = [0, 1]$ then $\min A = 0$

Existence of Max and Min

For infinite subsets of $\mathbb{R}$, max and min may not exist

Example. $\max \mathbb{N}$ does not exist

Suppose to the contrary that $n^* = \max \mathbb{N}$

By the definition of the maximum, $n^* \in \mathbb{N}$

Now consider

$$n^{**} := n^* + 1$$

Clearly

$$n^{**} \in \mathbb{N} \quad \text{and} \quad n^{**} > n^*$$

This contradicts the definition of n^*

Example. $\max(0,1)$ does not exist

Suppose to the contrary that $a^* = \max(0,1)$

By the definition of the maximum, $a^* \in (0,1)$

Hence $a^* < 1$

Now consider

$$a^{**} := (1 + a^*)/2$$

Clearly

$$a^{**} \in (0,1) \text{ and } a^{**} > a^*$$

Contradicts hypothesis that a^* is the maximum

Max/Min vs Sup/Inf

When max and min exist they agree with sup and inf

Facts Let A be any subset of $\mathbb{R}$

1. If $\sup A \in A$, then $\max A$ exists and $\max A = \sup A$
2. If $\inf A \in A$, then $\min A$ exists and $\min A = \inf A$

Proof of case 1: Let $a^* := \sup A$ and suppose $a^* \in A$

We want to show that $\max A = a^*$

Since $a^* \in A$, we need only show that $a \leq a^*$ for all $a \in A$

This follows from $a^* = \sup A$, which implies $a^* \in U(A)$

Existence of Max and Min for Sets

Fact. If $F \subset \mathbb{R}$ is a closed and bounded, then $\max F$ and $\min F$ both exist

Proof for the max case:

Since F is bounded,

- $\sup F$ exists
- $\exists$ a sequence $\{x_n\} \subset F$ with $x_n \to \sup F$

Since F is closed, this implies that $\sup F \in F$

Hence $\max F$ exists and $\max F = \sup F$

Optimizing Functions

Now we turn to extrema (sup / max / etc.) for functions

This is not a new concept — it's just about extrema of sets

...but the sets are the range of functions

In particular

- The sup of a function f is just the sup of its range
- The max of a function f is just the max of its range
- etc.

Througout we use the notation

$$f(A) := \{f(\mathbf{x}) : \mathbf{x} \in A\}$$

Sup and Inf for Functions

Let $f \colon A \to \mathbb{R}$, where A is any set

The **supremum of f on A** is defined as

$$\sup_{\mathbf{x} \in A} f(\mathbf{x}) := \sup f(A)$$

The **infimum of f on A** is defined as

$$\inf_{\mathbf{x} \in A} f(\mathbf{x}) := \inf f(A)$$

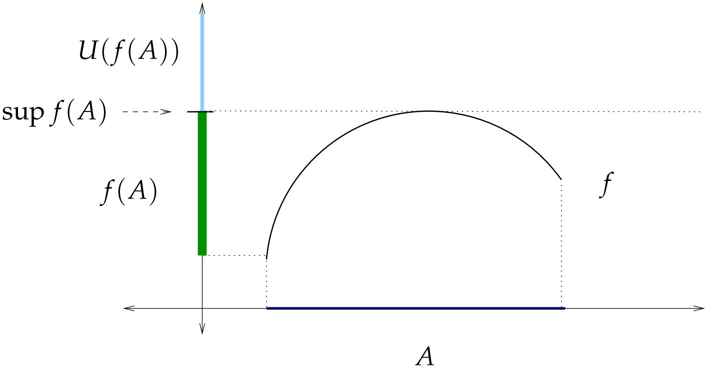


Figure : The supremum of f on A

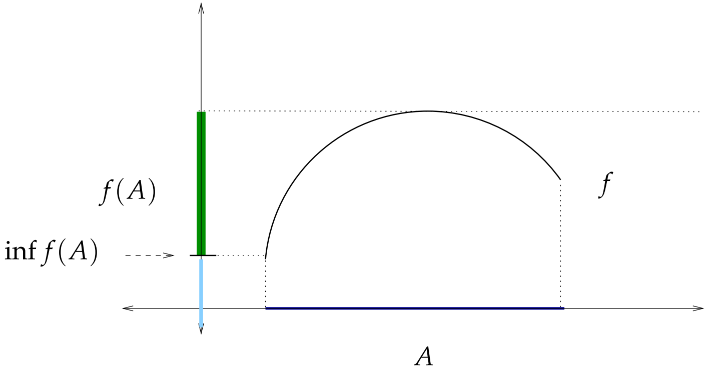


Figure : The infimum of f on A

Max and Min for Functions

Let $f\colon A \to \mathbb{R}$ where A is any set

The **maximum** of f on A is defined as

$$\max_{\mathbf{x}\in A} f(\mathbf{x}) := \max f(A)$$

The **minimum** of f on A is defined as

$$\min_{\mathbf{x}\in A} f(\mathbf{x}) := \min f(A)$$

A **maximizer** of f on A is a point $\mathbf{a}^* \in A$ such that

$$f(\mathbf{a}^*) = \max_{\mathbf{x} \in A} f(\mathbf{x})$$

Equivalent:

$$\mathbf{a}^* \in A \text{ and } f(\mathbf{a}^*) \geq f(\mathbf{x}) \text{ for all } \mathbf{x} \in A$$

The set of all maximizers denoted by

$$\operatorname*{argmax}_{\mathbf{x} \in A} f(\mathbf{x})$$

A **minimizer** of f on A is a point $\mathbf{a}^* \in A$ such that

$$f(\mathbf{a}^*) = \min_{\mathbf{x} \in A} f(\mathbf{x})$$

Equivalent:

$$\mathbf{a}^* \in A \text{ and } f(\mathbf{a}^*) \leq f(\mathbf{x}) \text{ for all } \mathbf{x} \in A$$

The set of all minimizers denoted by

$$\operatorname*{argmin}_{\mathbf{x} \in A} f(\mathbf{x})$$

Now we come to the famous **Weierstrass extreme value theorem**

Fact. If f is continuous and A is closed and bounded, then f has both a maximizer and a minimizer in A

Proof sketch for the max case:

Can show under the assumptions that $f(A)$ is closed and bounded

- proof uses Bolzano–Weierstrass theorem, details omitted

Hence the max of $f(A)$ exists, and we can write

$$M^* := \max f(A) := \max\{f(\mathbf{x}) : \mathbf{x} \in A\}$$

The point $\mathbf{x}^* \in A$ such that $f(\mathbf{x}^*) = M^*$ is a maximizer

Example. Consider the problem

$$\max_{c_1, c_2} U(c_1, c_2) := \sqrt{c_1} + \beta\sqrt{c_2}$$

$$\text{s.t. } c_2 \leq (1+r)(w - c_1), \quad c_i \geq 0 \text{ for } i = 1, 2$$

where

- r = interest rate, w = wealth, β = discount factor
- all parameters > 0

Let B be all (c_1, c_2) satisfying the constraint

Ex. Show that the budget set B is a closed, bounded subset of $\mathbb{R}^2$

Ex. Show that U is continuous on B

We conclude that a maximizer exists

Properties of Optima

We now state some useful facts regarding optima

Sometimes we state properties about sups and infs

- rather than max and min

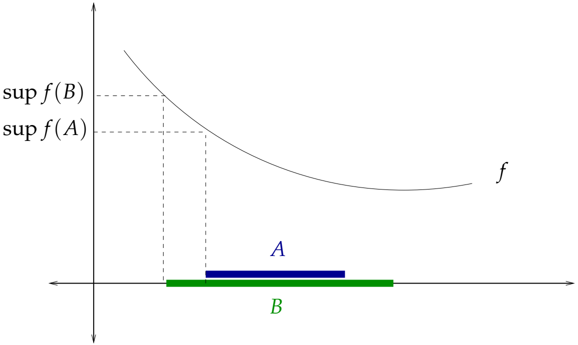
This is so we don't have to keep saying "if it exsits"

But remember that if it does exist then the same properties apply

- if a max exists, then it's a sup, etc.

Fact. If $A \subset B$ and $f\colon B \to \mathbb{R}$, then

$$\sup_{\mathbf{x}\in A} f(\mathbf{x}) \le \sup_{\mathbf{x}\in B} f(\mathbf{x}) \qquad \text{and} \qquad \inf_{\mathbf{x}\in A} f(\mathbf{x}) \ge \inf_{\mathbf{x}\in B} f(\mathbf{x})$$



Proof, for the sup case:

Let A, B and f be as in the statement of the fact

We already know that $C \subset D \implies \sup C \leq \sup D$

Hence it suffices to show that $f(A) \subset f(B)$, because then

$$\sup_{\mathbf{x} \in A} f(\mathbf{x}) := \sup f(A) \leq \sup f(B) =: \sup_{\mathbf{x} \in B} f(\mathbf{x})$$

To see that $f(A) \subset f(B)$, take any $y \in f(A)$

By definition, $\exists\, \mathbf{x} \in A$ such that $f(\mathbf{x}) = y$

Since $A \subset B$ we must have $\mathbf{x} \in B$

So $f(\mathbf{x}) = y$ for some $\mathbf{x} \in B$, and hence $y \in f(B)$

Thus $f(A) \subset f(B)$ as was to be shown

Example. "If you have more choice then you're better off"

Consider the problem of maximizing utility

$$U(x_1, x_2) = \alpha \log(x_1) + \beta \log(x_2)$$

over all (x_1, x_2) in the budget set

$$B(m) := \left\{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_i > 0 \text{ and } p_1 x_1 + p_2 x_2 \leq m\right\}$$

Thus, we solve

$$\max_{\mathbf{x} \in B(m)} U(\mathbf{x})$$

Clearly $m \leq m' \implies B(m) \subset B(m')$

Hence the maximal value goes up as m increases

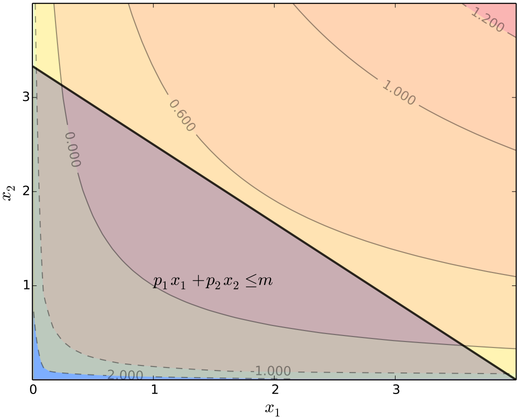


Figure : Budget set $B(m)$

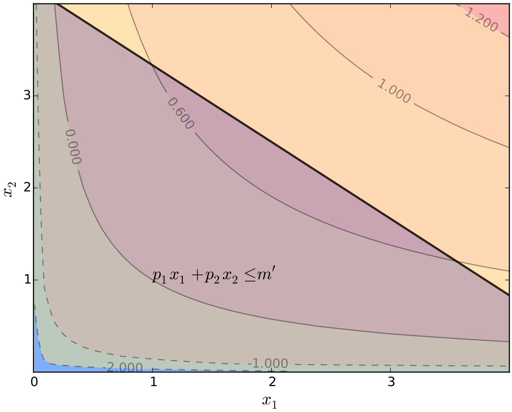


Figure : Budget set $B(m')$

Example. Let y_n be income and x_n be years education

Consider regressing income on education:

$$y_n = \alpha + \beta x_n + \epsilon_n$$

We have data for $n = 1, \ldots, N$ individuals

Successful regression is often associated with large R^2

- A measure of "goodness of fit"

Large R^2 occurs when we have a small sum of squared residuals

$$\mathrm{ssr}_a := \min_{\alpha,\beta} \sum_{n=1}^{N} (y_n - \alpha - \beta x_n)^2$$

However, we can always reduce the ssr by including irrelevant variables

- e.g., $z_n =$ consumption of bacon in kgs per annum

$$\mathrm{ssr}_b := \min_{\alpha,\beta,\gamma} \sum_{n=1}^{N} (y_n - \alpha - \beta x_n - \gamma z_n)^2 \leq \mathrm{ssr}_a$$

Proof: Let

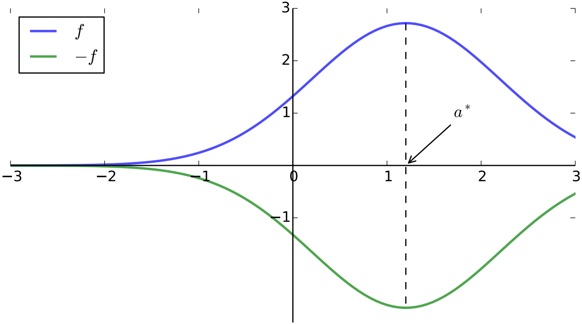
$$\boldsymbol{\theta} := (\alpha, \beta, \gamma), \qquad f(\boldsymbol{\theta}) := \sum_{n=1}^{N} (y_n - \alpha - \beta x_n - \gamma z_n)^2$$

Then

$$\mathrm{ssr}_b = \min_{\boldsymbol{\theta} \in \mathbb{R}^3} f(\boldsymbol{\theta}) \leq \min_{\substack{\boldsymbol{\theta} \in \mathbb{R}^3 \\ \gamma = 0}} f(\boldsymbol{\theta}) = \mathrm{ssr}_a$$

Fact. If $f\colon A \to \mathbb{R}$, then

$$\mathbf{a}^* \in \operatorname*{argmax}_{\mathbf{x}\in A} f(\mathbf{x}) \iff \mathbf{a}^* \in \operatorname*{argmin}_{\mathbf{x}\in A} -f(\mathbf{x})$$



Proof: Let's prove that, when $g = -f$,

$$\mathbf{a}^* \in \operatorname*{argmax}_{\mathbf{x}\in A} f(\mathbf{x}) \implies \mathbf{a}^* \in \operatorname*{argmin}_{\mathbf{x}\in A} g(\mathbf{x})$$

To begin, let $\mathbf{a}^*$ be a maximizer of f on A

Then, for any given $\mathbf{x} \in A$ we have $f(\mathbf{a}^*) \geq f(\mathbf{x})$

$$\therefore \quad -f(\mathbf{a}^*) \leq -f(\mathbf{x})$$

$$\therefore \quad g(\mathbf{a}^*) \leq g(\mathbf{x})$$

Hence $\mathbf{a}^*$ is a minimizer of g on A

- because the last inequality was shown for any $\mathbf{x} \in A$

Example. Most numerical routines provide minimization only

Suppose we want to maximize $f(x) = 3 \ln x - x$ on $(0, \infty)$

We can do this by finding the minimizer of $-f$

```
In [1]: from scipy.optimize import fminbound
In [2]: import numpy as np

In [3]: f = lambda x: 3 * np.log(x) - x
In [4]: g = lambda x: -f(x)  # Find min of -f

In [5]: fminbound(g, 1, 100)
Out[5]: 3.0000015012062393
```

Given $A \subset \mathbb{R}^K$, let

- $f \colon A \to B \subset \mathbb{R}$
- $h \colon B \to \mathbb{R}$ and $g := h \circ f$

Fact. If h is strictly increasing, then

$$\operatorname*{argmax}_{\mathbf{x} \in A} f(\mathbf{x}) = \operatorname*{argmax}_{\mathbf{x} \in A} g(\mathbf{x})$$

Proof of $\subset$: Let $\mathbf{a}^* \in \operatorname{argmax}_{\mathbf{x} \in A} f(\mathbf{x})$

If $\mathbf{x} \in A$, then $f(\mathbf{x}) \leq f(\mathbf{a}^*)$, and hence $h(f(\mathbf{x})) \leq h(f(\mathbf{a}^*))$

In other words, $g(\mathbf{x}) \leq g(\mathbf{a}^*)$ for any $\mathbf{x} \in A$

Hence $\mathbf{a}^* \in \operatorname{argmax}_{\mathbf{x} \in A} g(\mathbf{x})$ as claimed

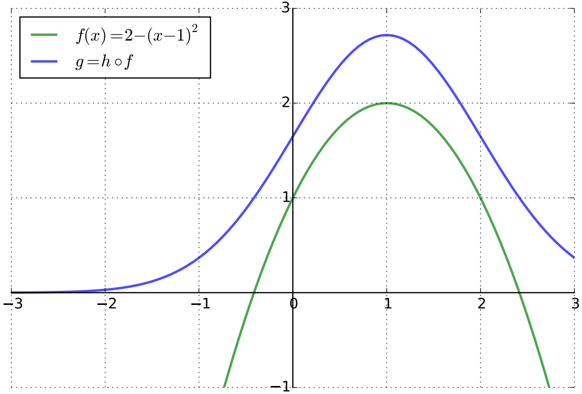


Figure : Increasing transform $h(x) = \exp(x/2)$ preserves the maximizer

Example. A well known statistical problem is to maximize the exponential likelihood function:

$$\max_{\lambda>0} L(\lambda) \quad \text{where} \quad L(\lambda) := \lambda^N \exp\left(-\lambda \sum_{n=1}^{N} x_n\right)$$

It's easier to maximize the log-likelihood function

$$\ell(\lambda) := \log(L(\lambda)) = N \log(\lambda) - \lambda \sum_{n=1}^{N} x_n$$

The unique solution

$$\hat{\lambda} := \frac{N}{\sum_{n=1}^{N} x_n}$$

is also the unique maximiser of $L(\lambda)$

In the next few slides

1. A is any set
2. f is some function from A to $\mathbb{R}$
3. g is some function from A to $\mathbb{R}$

To simplify notation, we define

$$\inf f := \inf_{\mathbf{x} \in A} f(\mathbf{x})$$

and

$$\sup f := \sup_{\mathbf{x} \in A} f(\mathbf{x})$$

Fact.

$$f(\mathbf{x}) \leq g(\mathbf{x}) \text{ for all } \mathbf{x} \in A \implies \sup f \leq \sup g$$

Proof: Fix any such functions f and g and any $\mathbf{x} \in A$

We have

$$f(\mathbf{x}) \leq g(\mathbf{x}) \leq \sup g$$

Hence $\sup g$ is an upper bound for $\{f(\mathbf{x}) : \mathbf{x} \in A\}$

Since the supremum is the least upper bound, this gives

$$\sup f \leq \sup g$$

Fact.

$$\sup_{\mathbf{x} \in A}(f(\mathbf{x}) + g(\mathbf{x})) \leq \sup_{\mathbf{x} \in A} f(\mathbf{x}) + \sup_{\mathbf{x} \in A} g(\mathbf{x})$$

Proof: Fix any such functions f and g and any $\mathbf{x} \in A$

We have

$$f(\mathbf{x}) \leq \sup f \quad \text{and} \quad g(\mathbf{x}) \leq \sup g$$

$$\therefore \quad f(\mathbf{x}) + g(\mathbf{x}) \leq \sup f + \sup g$$

$$\therefore \quad \sup(f + g) \leq \sup f + \sup g$$

Fact.

$$\left|\sup_{\mathbf{x}\in A} f(\mathbf{x}) - \sup_{\mathbf{x}\in A} g(\mathbf{x})\right| \leq \sup_{\mathbf{x}\in A} |f(\mathbf{x}) - g(\mathbf{x})|$$

Proof: Picking any such f, g, we have

$$\begin{aligned} \sup f = \sup(f - g + g) &\leq \sup(f - g) + \sup g \\ &\leq \sup |f - g| + \sup g \end{aligned}$$

$$\therefore\ \sup f - \sup g \leq \sup |f - g|$$

Same argument reversing roles of f and g finishes the proof