ECON2125/4021/8013

Lecture 17

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Semester 1, 2015

Announcements: Midterm

- Some students did very well
- But many competent students did not

As a result marks have been scaled upwards

- No mark has decreased from scaling
- An order preserving transformation
- Undergrad and graduates treated separately

The marks you receive (tomorrow?) will be the scaled marks

Announcements: Extra Reading

The current section of the course is on analysis

If you want supplementary reading try

- Simon and Blume, **Mathematics for Economists**, Ch. 12
- Sundaram, **A First Course in Optimization Theory**, Appendix B, C

Perhaps useful but not required reading

Derivatives

Let $f\colon (a,b) \to \mathbb{R}$ and let $x \in (a,b)$

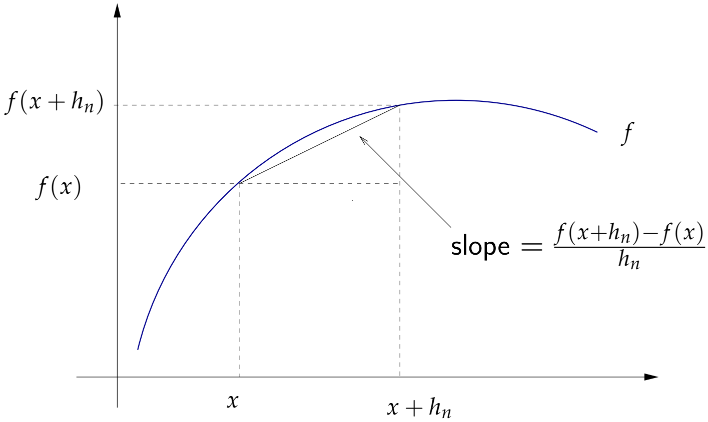
Let H be all sequences $\{h_n\}$ such that $h_n \neq 0$ and $h_n \to 0$

If there exists a constant $f'(x)$ such that

$$\frac{f(x+h_n) - f(x)}{h_n} \to f'(x)$$

for every $\{h_n\} \in H$, then

- f is said to be **differentiable** at x
- $f'(x)$ is called the **derivative** of f at x


$f(x+h_n)$
$f(x)$
f
slope $= \frac{f(x+h_n)-f(x)}{h_n}$
x
$x+h_n$

Example. Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$

Fix any $x \in \mathbb{R}$ and any $h_n \to 0$

We have

$$\frac{f(x+h_n) - f(x)}{h_n} = \frac{(x+h_n)^2 - x^2}{h_n}$$

$$= \frac{x^2 + 2xh_n + h_n^2 - x^2}{h_n} = 2x + h_n$$

$$\therefore \quad f'(x) = \lim_{n \to \infty} (2x + h_n) = 2x$$

Example. Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$

This function is not differentiable at $x = 0$

Indeed, if $h_n = 1/n$, then

$$\frac{f(0+h_n) - f(0)}{h_n} = \frac{|0+1/n| - |0|}{1/n} \to 1$$

On the other hand, if $h_n = -1/n$, then

$$\frac{f(0+h_n) - f(0)}{h_n} = \frac{|0-1/n| - |0|}{-1/n} \to -1$$

Just for intuition: Taylor series

Loosely speaking, if $f \colon \mathbb{R} \to \mathbb{R}$ is suitably differentiable at a, then

$$f(x) \approx f(a) + f'(a)(x-a)$$

for x very close to a,

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

on a slightly wider interval, etc.

These are the 1st and 2nd order **Taylor series approximations** to f at a respectively

As the order goes higher we get better approximation

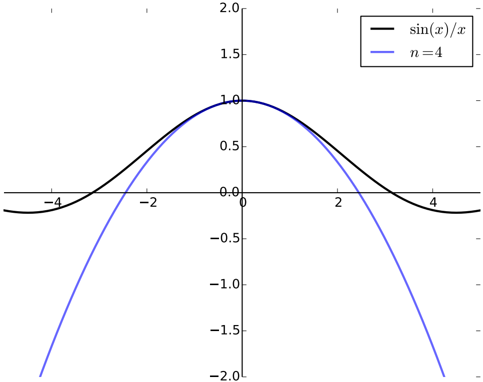


Figure : 4th order Taylor series for $f(x) = \sin(x)/x$ at 0

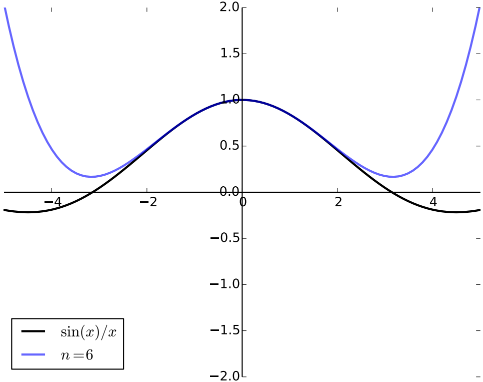


Figure : 6th order Taylor series for $f(x) = \sin(x)/x$ at 0

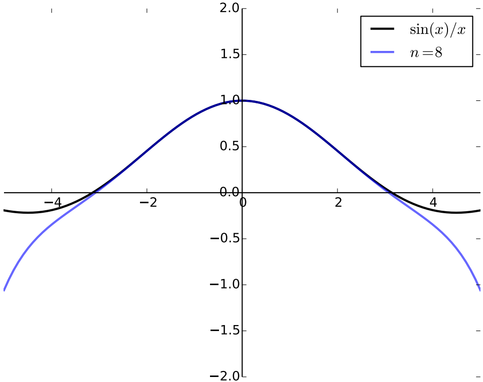


Figure : 8th order Taylor series for $f(x) = \sin(x)/x$ at 0

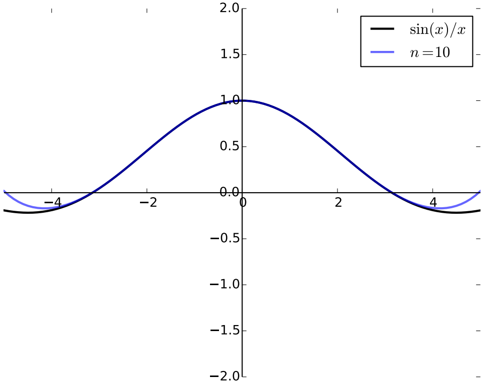


Figure : 10th order Taylor series for $f(x) = \sin(x)/x$ at 0

Analysis in $\mathbb{R}^K$

Now we switch from studying points $x \in \mathbb{R}$ to vectors $\mathbf{x} \in \mathbb{R}^K$

- Replace distance $|x - y|$ with $\|\mathbf{x} - \mathbf{y}\|$

Many of the same results go through otherwise unchanged

We state the analogous results briefly since

- You already have the intuition from $\mathbb{R}$
- Similar arguments, just replacing $|\cdot|$ with $\|\cdot\|$

We'll spend longer on things that are different

Bounded sets and ϵ-balls

A set $A \subset \mathbb{R}^K$ called **bounded** if

$$\exists\, M \in \mathbb{R} \text{ s.t. } \|\mathbf{x}\| \leq M, \quad \forall\, \mathbf{x} \in A$$

Remarks:

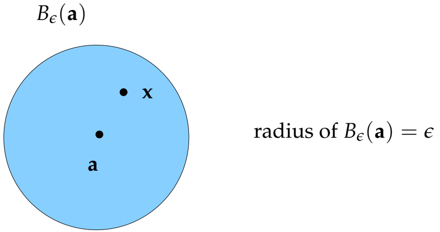
- A generalization of the scalar definition
- When $K = 1$, the norm $\| \cdot \|$ reduces to $| \cdot |$

Fact. If A and B are bounded sets then so is $C := A \cup B$

Proof: Same as the scalar case — just replace $| \cdot |$ with $\| \cdot \|$

Ex. Check it

For $\epsilon > 0$, the ϵ-ball $B_\epsilon(\mathbf{a})$ around $\mathbf{a} \in \mathbb{R}^K$ is all $\mathbf{x} \in \mathbb{R}^K$ such that $\|\mathbf{a} - \mathbf{x}\| < \epsilon$



Fact. If $\mathbf{x}$ is in every ϵ-ball around $\mathbf{a}$ then $\mathbf{x} = \mathbf{a}$

Fact. If $\mathbf{a} \neq \mathbf{b}$, then $\exists\, \epsilon > 0$ s.t. $B_\epsilon(\mathbf{a}) \cap B_\epsilon(\mathbf{b}) = \emptyset$

A **sequence** $\{\mathbf{x}_n\}$ in $\mathbb{R}^K$ is a function from $\mathbb{N}$ to $\mathbb{R}^K$

Sequence $\{\mathbf{x}_n\}$ said to **converge** to $\mathbf{a} \in \mathbb{R}^K$ if

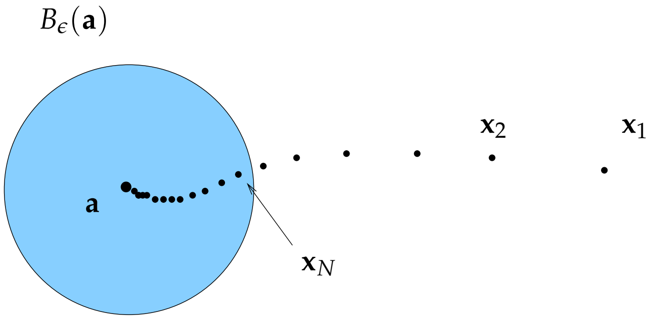
$$\forall \epsilon > 0, \; \exists \, N \in \mathbb{N} \;\; \text{s.t.} \; n \geq \mathbb{N} \implies \mathbf{x}_n \in B_\epsilon(\mathbf{a})$$

We say: "$\{\mathbf{x}_n\}$ eventually in any ϵ-neighborhood of $\mathbf{a}$"

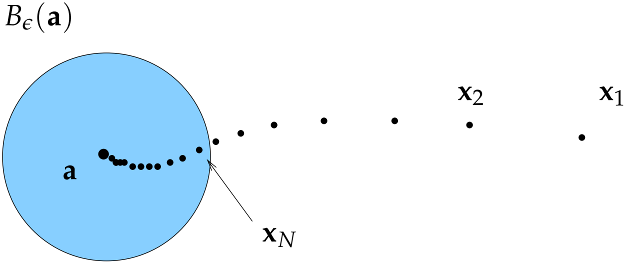
In this case $\mathbf{a}$ is called the **limit** of the sequence, and we write

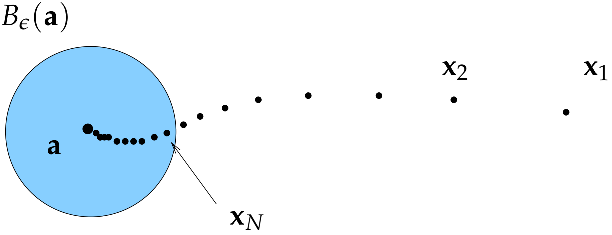
$$\mathbf{x}_n \to \mathbf{a} \;\; \text{as} \;\; n \to \infty \qquad \text{or} \qquad \lim_{n \to \infty} \mathbf{x}_n = \mathbf{a}$$

We call $\{\mathbf{x}_n\}$ **convergent** if it converges to some limit in $\mathbb{R}^K$



$B_\epsilon(\mathbf{a})$
$\mathbf{x}_2$
$\mathbf{x}_1$
$\mathbf{a}$
$\mathbf{x}_N$


$B_\epsilon(\mathbf{a})$
$\mathbf{x}_2$
$\mathbf{x}_1$
$\mathbf{a}$
$\mathbf{x}_N$


$B_\epsilon(\mathbf{a})$
$\mathbf{x}_2$
$\mathbf{x}_1$
$\mathbf{a}$
$\mathbf{x}_N$

Vector vs Componentwise Convergence

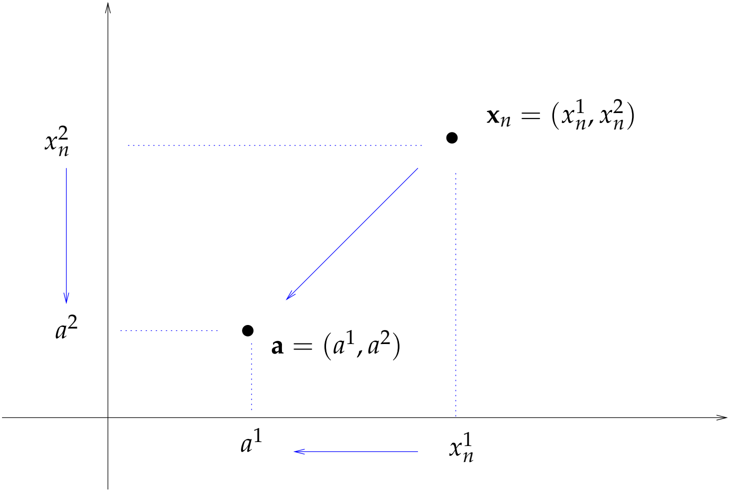
Fact. A sequence $\{\mathbf{x}_n\}$ in $\mathbb{R}^K$ converges to $\mathbf{a} \in \mathbb{R}^K$ if and only if each component sequence converges in $\mathbb{R}$

That is,

$$\begin{pmatrix} x_n^1 \\ \vdots \\ x_n^K \end{pmatrix} \to \begin{pmatrix} a^1 \\ \vdots \\ a^K \end{pmatrix} \quad \text{in } \mathbb{R}^K \qquad \iff \qquad \begin{matrix} x_n^1 \to a^1 & \text{in } \mathbb{R} \\ \vdots & \\ x_n^K \to a^K & \text{in } \mathbb{R} \end{matrix}$$

Equivalent:

$$\mathbf{x}_n \to \mathbf{a} \ \text{ in } \mathbb{R}^K \qquad \iff \qquad \mathbf{e}_k' \mathbf{x}_n \to \mathbf{e}_k' \mathbf{a} \text{ in } \mathbb{R} \text{ for all } k$$


$\mathbf{x}_n = (x_n^1, x_n^2)$
x_n^2
a^2
$\mathbf{a} = (a^1, a^2)$
a^1
x_n^1

From Scalar to Vector Analysis

More definitions analogous to scalar case:

A sequence $\{\mathbf{x}_n\}$ is called **Cauchy** if

$$\forall \, \epsilon > 0, \;\; \exists \, N \in \mathbb{N} \;\; \text{s.t.} \;\; n, m \geq N \implies \|\mathbf{x}_n - \mathbf{x}_m\| < \epsilon$$

A sequence $\{\mathbf{x}_{n_k}\}$ is called a **subsequence** of $\{\mathbf{x}_n\}$ if

1. $\{\mathbf{x}_{n_k}\}$ is a subset of $\{\mathbf{x}_n\}$
2. the indices n_k are strictly increasing

Facts Analogous to the scalar case,

1. $\mathbf{x}_n \to \mathbf{a}$ in $\mathbb{R}^K$ if and only if $\|\mathbf{x}_n - \mathbf{a}\| \to 0$ in $\mathbb{R}$
2. If $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$ then $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$
3. If $\mathbf{x}_n \to \mathbf{x}$ and $\alpha \in \mathbb{R}$ then $\alpha \mathbf{x}_n \to \alpha \mathbf{x}$
4. If $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{z} \in \mathbb{R}^K$ then $\mathbf{z}'\mathbf{x}_n \to \mathbf{z}'\mathbf{x}$
5. Each sequence in $\mathbb{R}^K$ has at most one limit
6. Every convergent sequence in $\mathbb{R}^K$ is bounded
7. Every convergent sequence in $\mathbb{R}^K$ is Cauchy
8. Every Cauchy sequence in $\mathbb{R}^K$ is convergent

Ex. Adapt proofs given for the scalar case to these results

Example. Let's check that

$$\mathbf{x}_n \to \mathbf{a} \text{ in } \mathbb{R}^K \iff \|\mathbf{x}_n - \mathbf{a}\| \to 0 \text{ in } \mathbb{R}$$

- $\mathbf{x}_n \to \mathbf{a}$ in $\mathbb{R}^K$ means that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq \mathbb{N} \implies \|\mathbf{x}_n - \mathbf{a}\| < \epsilon$$

- $\|\mathbf{x}_n - \mathbf{a}\| \to 0$ in $\mathbb{R}$ means that

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } n \geq \mathbb{N} \implies |\|\mathbf{x}_n - \mathbf{a}\| - 0| < \epsilon$$

Obviously equivalent

Reminder — these **Facts** are more general than scalar ones

- True for any finite K
- So true for $K = 1$
- This recovers the corresponding scalar fact

You can forget the scalar fact if you remember the vector one

Infinite Sums in $\mathbb{R}^K$

Analogous to the scalar case, an infinite sum in $\mathbb{R}^K$ is the limit of the partial sum:

- If $\{\mathbf{x}_n\}$ is a sequence in $\mathbb{R}^K$, then

$$\sum_{n=1}^{\infty} \mathbf{x}_n := \lim_{J \to \infty} \sum_{n=1}^{J} \mathbf{x}_n \text{ if the limit exists}$$

In other words,

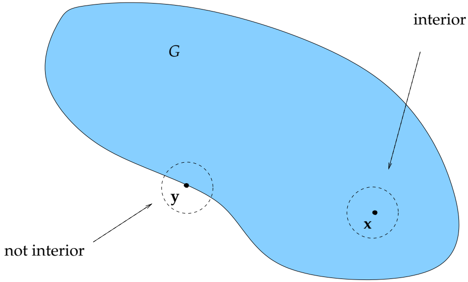
$$\mathbf{y} = \sum_{n=1}^{\infty} \mathbf{x}_n \quad \iff \quad \lim_{J \to \infty} \left\| \sum_{n=1}^{J} \mathbf{x}_n - \mathbf{y} \right\| \to 0$$

Open Sets

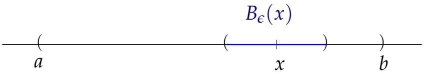
Let $G \subset \mathbb{R}^K$

We call $\mathbf{x} \in G$ **interior** to G if $\exists\ \epsilon > 0$ with $B_\epsilon(\mathbf{x}) \subset G$

Loosely speaking, interior means "not on the boundary"



Example. If $G = (a, b)$ for some $a < b$, then any $x \in (a, b)$ is interior



Proof: Fix any $a < b$ and any $x \in (a, b)$

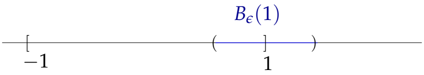
Let $\epsilon := \min\{x - a, b - x\}$

If $y \in B_\epsilon(x)$ then $y < b$ because

$$y = y + x - x \leq |y - x| + x < \epsilon + x \leq b - x + x = b$$

Ex. Show $y \in B_\epsilon(x) \implies y > a$

Example. If $G = [-1, 1]$, then 1 is not interior



Intuitively, any ϵ-ball centered on 1 will contain points > 1

More formally, pick any $\epsilon > 0$ and consider $B_\epsilon(1)$

There exists a $y \in B_\epsilon(1)$ such that $y \notin [-1, 1]$

For example, consider the point $y := 1 + \epsilon/2$

Ex. Check this point

- lies in $B_\epsilon(1)$
- but not in $[-1, 1]$

A set $G \subset \mathbb{R}^K$ is called **open** if all of its points are interior

Example. Any "open" interval $(a, b) \subset \mathbb{R}$, since we showed all points are interior

Other Examples.

- any "open" ball $B_\epsilon(\mathbf{a}) = \{\mathbf{x} \in \mathbb{R}^K : \|\mathbf{x} - \mathbf{a}\| < \epsilon\}$
- $\mathbb{R}^K$ itself

Examples. of sets that are not open

- $(a, b]$ because b is not interior
- $[a, b)$ because a is not interior

Closed Sets

A set $F \subset \mathbb{R}^K$ is called **closed** if every convergent sequence in F converges to a point in F

Rephrased: If $\{\mathbf{x}_n\} \subset F$ and $\mathbf{x}_n \to \mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^K$, then $\mathbf{x} \in F$

Example. All of $\mathbb{R}^K$ is closed because every sequence converging to a point in $\mathbb{R}^K$ converges to a point in $\mathbb{R}^K$... right?

Example. If $(-1, 1) \subset \mathbb{R}$ is **not** closed

Proof: True because

1. $x_n := 1 - 1/n$ is a sequence in $(-1, 1)$ converging to 1,
2. and yet $1 \notin (-1, 1)$

Example. If $F = [a, b] \subset \mathbb{R}$ then F is closed in $\mathbb{R}$

Proof: Take any sequence $\{x_n\}$ such that

- $x_n \in F$ for all n
- $x_n \to x$ for some $x \in \mathbb{R}$

We claim that $x \in F$

Recall that (weak) inequalities are preserved under limits:

- $x_n \leq b$ for all n and $x_n \to x$, so $x \leq b$
- $x_n \geq a$ for all n and $x_n \to x$, so $x \geq a$

$$\therefore \quad x \in [a, b] =: F$$

Example. Any "hyperplane" of the form

$$H = \{\mathbf{x} \in \mathbb{R}^K : \mathbf{x}'\mathbf{a} = c\}$$

is closed

Proof: Fix $\mathbf{a} \in \mathbb{R}^K$ and $c \in \mathbb{R}$ and let H be as above

Let $\{\mathbf{x}_n\} \subset H$ with $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^K$

We claim that $\mathbf{x} \in H$

Since $\mathbf{x}_n \in H$ and $\mathbf{x}_n \to \mathbf{x}$ we have

$$\mathbf{x}_n'\mathbf{a} \to \mathbf{x}'\mathbf{a} \text{ in } \mathbb{R} \quad \text{and} \quad \mathbf{x}_n'\mathbf{a} = c \text{ for all } n$$

$$\therefore \quad \mathbf{x}'\mathbf{a} = \lim_n \mathbf{x}_n'\mathbf{a} = \lim_n c = c$$

$$\therefore \quad \mathbf{x} \in H$$

Properties of Open and Closed Sets

Fact. $G \subset \mathbb{R}^K$ is open $\iff$ G^c is closed

Proof: Let's just check $\implies$

Pick any G and let $F := G^c$

Suppose to the contrary that G is open but F is not closed, so

$$\exists \text{ a sequence } \{\mathbf{x}_n\} \subset F \text{ with limit } \mathbf{x} \notin F$$

Then $\mathbf{x} \in G$, and since G open, $\exists\, \epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset G$

Since $\mathbf{x}_n \to \mathbf{x}$ we can choose an $N \in \mathbb{N}$ with $\mathbf{x}_N \in B_\epsilon(\mathbf{x})$

This contradicts $\mathbf{x}_n \in F$ for all n

Facts

1. Any union of open sets is open
2. Any intersection of closed sets is closed

Proof of first fact:

Let $G := \cup_{\lambda \in \Lambda} G_\lambda$, where each G_λ is open

We claim that any given $\mathbf{x} \in G$ is interior to G

Pick any $\mathbf{x} \in G$

By definition, $\mathbf{x} \in G_\lambda$ for some λ

Since G_λ is open, $\exists \, \epsilon > 0$ such that $B_\epsilon(\mathbf{x}) \subset G_\lambda$

But $G_\lambda \subset G$, so $B_\epsilon(\mathbf{x}) \subset G$ also holds

In other words, $\mathbf{x}$ is interior to G

Continuity

One of the most fundamental properties of functions

Related to existence of

- optima
- roots
- fixed points
- etc

as well as a variety of other useful concepts

Let $F \colon A \to \mathbb{R}^J$ where A is a subset of $\mathbb{R}^K$

F is called **continuous at** $\mathbf{x} \in A$ if

$$\mathbf{x}_n \to \mathbf{x} \quad \implies \quad F(\mathbf{x}_n) \to F(\mathbf{x})$$

Requires that

- $F(\mathbf{x}_n)$ converges for each choice of $\mathbf{x}_n \to \mathbf{x}$,
- The limit is always the same, and that limit is $F(\mathbf{x})$

F is called **continuous** if it is continuous at every $\mathbf{x} \in A$

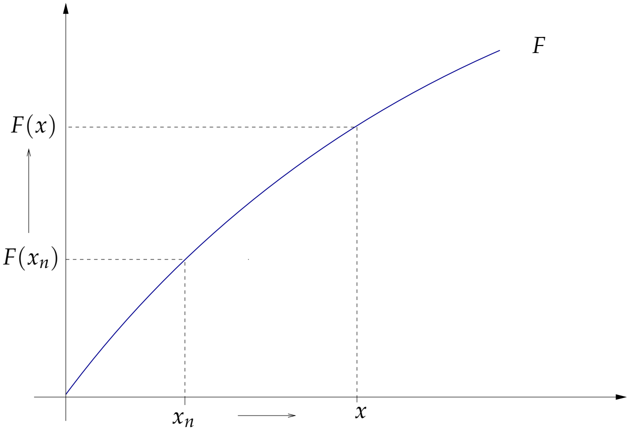


Figure : Continuity

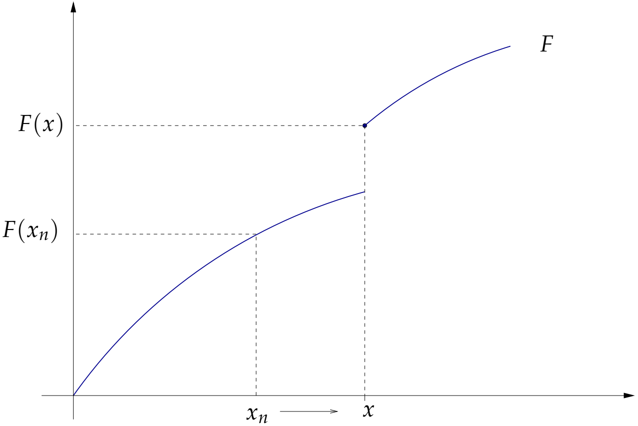


Figure : Discontinuity at x

Example. Let $\mathbf{A}$ be an $J \times K$ matrix and let $F(\mathbf{x}) = \mathbf{A}\mathbf{x}$

The function F is continuous at every $\mathbf{x} \in \mathbb{R}^K$

To see this take

- any $\mathbf{x} \in \mathbb{R}^K$
- any $\mathbf{x}_n \to \mathbf{x}$

By the definition of the matrix norm $\|\mathbf{A}\|$, we have

$$\|\mathbf{A}\mathbf{x}_n - \mathbf{A}\mathbf{x}\| = \|\mathbf{A}(\mathbf{x}_n - \mathbf{x})\| \leq \|\mathbf{A}\| \|\mathbf{x}_n - \mathbf{x}\|$$

$$\therefore \quad \mathbf{x}_n \to \mathbf{x} \implies \mathbf{A}\mathbf{x}_n \to \mathbf{A}\mathbf{x}$$

Exactly what rules are we using here?

Some functions known to be continuous on their domains:

- $x \mapsto x^\alpha$
- $x \mapsto |x|$
- $x \mapsto \log(x)$
- $x \mapsto \exp(x)$
- $x \mapsto \sin(x)$
- $x \mapsto \cos(x)$
- etc

Discontinuous at zero: $x \mapsto \mathbb{1}\{x > 0\}$

Let F and G be functions and let $\alpha \in \mathbb{R}$

Facts

1. If F and G are continuous at $\mathbf{x}$ then so is $F + G$, where

$$(F + G)(\mathbf{x}) := F(\mathbf{x}) + G(\mathbf{x})$$

2. If F is continuous at $\mathbf{x}$ then so is αF, where

$$(\alpha F)(\mathbf{x}) := \alpha F(\mathbf{x})$$

3. If F and G are continuous at $\mathbf{x}$ and real valued then so is FG, where

$$(FG)(\mathbf{x}) := F(\mathbf{x}) \cdot G(\mathbf{x})$$

In the latter case, if in addition $G(\mathbf{x}) \neq 0$, then F/G is also continuous

As a result, set of continuous functions is “closed” under elementary arithmetic operations

Example. The function $F\colon \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = \frac{\exp(x) + \sin(x)}{2 + \cos(x)} + \frac{x^4}{2} - \frac{\cos^3(x)}{8!}$$

is continuous

Proof: Just repeatedly apply the rules on the previous slide

Let's just check that

$$F \text{ and } G \text{ continuous at } \mathbf{x} \implies F + G \text{ continuous at } \mathbf{x}$$

Proof: Let F and G be continuous at $\mathbf{x}$

Pick any $\mathbf{x}_n \to \mathbf{x}$

We claim that $F(\mathbf{x}_n) + G(\mathbf{x}_n) \to F(\mathbf{x}) + G(\mathbf{x})$

By assumption, $F(\mathbf{x}_n) \to F(\mathbf{x})$ and $G(\mathbf{x}_n) \to G(\mathbf{x})$

From this and the triangle inequality we get

$$\|F(\mathbf{x}_n) + G(\mathbf{x}_n) - (F(\mathbf{x}) + G(\mathbf{x}))\|$$

$$\leq \|F(\mathbf{x}_n) - F(\mathbf{x})\| + \|G(\mathbf{x}_n) - G(\mathbf{x})\| \to 0$$

Order

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^K$

We write $\mathbf{x} \leq \mathbf{y}$ if every element is correspondingly ordered

Examples.

$$\begin{pmatrix} 1 \\ -2 \end{pmatrix} \leq \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{but} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix} \not\leq \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Letting $\mathbf{e}_k$ be the k-th canonical basis vector,

$$\mathbf{x} \leq \mathbf{y} \quad \iff \quad \mathbf{e}_k' \mathbf{x} \leq \mathbf{e}_k' \mathbf{y} \text{ in } \mathbb{R} \text{ for all } k$$

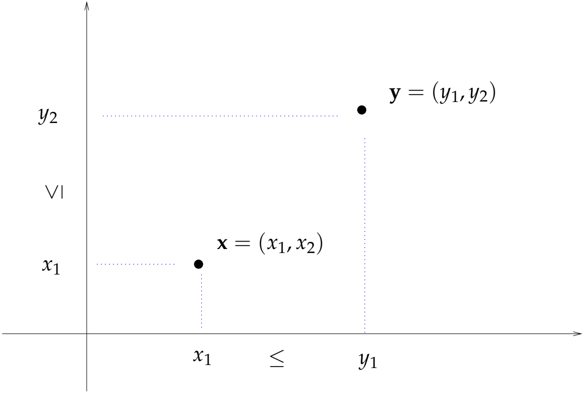


Figure : In $\mathbb{R}^2$, $\mathbf{x} \leq \mathbf{y}$ means $\mathbf{y}$ is north east of $\mathbf{x}$

Fact. If $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$ and $\mathbf{x}_n \leq \mathbf{y}_n$ for all $n \in \mathbb{N}$, then $\mathbf{x} \leq \mathbf{y}$

- extends scalar result to the vector case

Proof: Assume that $\mathbf{x}_n \to \mathbf{x}$, $\mathbf{y}_n \to \mathbf{y}$ and $\mathbf{x}_n \leq \mathbf{y}_n$ for all n

The claim is that $\mathbf{e}_k'\mathbf{x} \leq \mathbf{e}_k'\mathbf{y}$ for any k

Fix k in $1, \ldots, K$ and note that

- $\mathbf{e}_k'\mathbf{x}_n \to \mathbf{e}_k'\mathbf{x}$ (because $\mathbf{x}_n \to \mathbf{x}$)
- $\mathbf{e}_k'\mathbf{y}_n \to \mathbf{e}_k'\mathbf{y}$ (because $\mathbf{y}_n \to \mathbf{y}$)
- $\mathbf{e}_k'\mathbf{x}_n \leq \mathbf{e}_k'\mathbf{y}_n$ for all n (because $\mathbf{x}_n \leq \mathbf{y}_n$ for all n)

Hence, by the corresponding scalar result, $\mathbf{e}_k'\mathbf{x} \leq \mathbf{e}_k'\mathbf{y}$

A function $F\colon \mathbb{R}^K \to \mathbb{R}^J$ is called **increasing** if

$$\mathbf{x} \leq \mathbf{y} \implies F(\mathbf{x}) \leq F(\mathbf{y})$$

If $K = J = 1$, then this is the usual notion — graph of the function goes up (weakly)

Examples.

- $f(x) = x + c$ for any constant c
- $f(x) = cx$ for any $c \geq 0$
- $f(x) = \log(x)$ over $x \in (0, \infty)$
- $f(x) = x^c$ for any $c \geq 0$ over $x \in [0, \infty)$

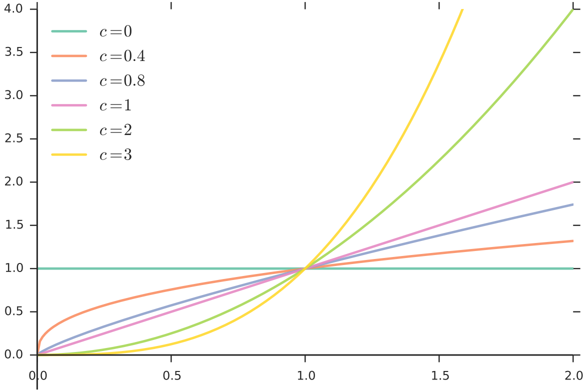


Figure : The function $f(x) = x^c$ on $[0, \infty)$ for different c

Example. If $\mathbf{a} \in \mathbb{R}^K$ satisfies $\mathbf{a} \geq \mathbf{0}$, then $f \colon \mathbb{R}^K \to \mathbb{R}$ defined by

$$f(\mathbf{x}) = \mathbf{a}'\mathbf{x}$$

is increasing

Proof: Pick any $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^K$ with $\mathbf{x} \leq \mathbf{y}$

By assumption, a_k is nonnegative and $x_k \leq y_k$ for all k

$$\therefore \quad f(\mathbf{x}) = \mathbf{a}'\mathbf{x} = \sum_{k=1}^K a_k x_k \leq \sum_{k=1}^K a_k y_k = f(\mathbf{y})$$

Ex. Letting $\mathbf{A}$ be any matrix, show that if all elements of $\mathbf{A}$ are nonnegative, then $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ is increasing