# ECON2125/4021/8013

## Lecture 15

John Stachurski

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# Announcements

- This week's Thursday lecture will be shifted to Friday
  - 9am on 23/04/2015 to 10am on 24/04/2015
  - Same location
  - To let people focus on exam preparation
- Preliminary date for final exam is June 11
  - Still subject to change

# Convergence in Distribution

Let

- $\{F_n\}_{n=1}^{\infty}$ be a sequence of cdfs
- $F$ be any cdf

We say that $\{F_n\}_{n=1}^{\infty}$ **converges weakly** to $F$ if

$$F_n(x) \to F(x) \quad \text{as} \quad n \to \infty$$

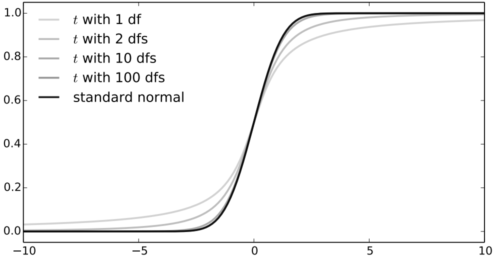
for any $x$ such that $F$ is continuous at $x$

- In essence, $F_n$ gets close to $F$ when $n$ is large

Example. **Student's t-density** with $n$ degrees of freedom is

$$p_n(x) := \frac{\Gamma(\frac{n+1}{2})}{(n\pi)^{1/2}\Gamma(\frac{n}{2})} \left(1 + n^{-1}x^2\right)^{-(n+1)/2}$$

It's well known that the corresponding cdfs $F_n$ converge weakly to the standard normal cdf



We say that $\{X_n\}_{n=1}^{\infty}$ converges to $X$ **in distribution** if

1. $X_n \sim F_n$,
2. $X \sim F$ and
3. $F_n \to F$ weakly

In this case we write $X_n \xrightarrow{d} X$

- In short, the distribution of $X_n$ converges to that of $X$

**Fact.** If $X_n \xrightarrow{p} X$, then $X_n \xrightarrow{d} X$

Example. If $X$ is any RV and $X_n := X + \frac{1}{n}$ then $X_n \xrightarrow{d} X$

Proof: Let $F$ and $F_n$ be the cdfs of $X$ and $X_n$ respectively

Observe that, $\forall\, x \in \mathbb{R}$,

$$F_n(x) = \mathbb{P}\left\{X + \frac{1}{n} \leq x\right\} = \mathbb{P}\left\{X \leq x - \frac{1}{n}\right\} = F\left(x - \frac{1}{n}\right)$$

Suppose that $F$ is continuous at $x$

Since $x - \frac{1}{n} \to x$, we have

$$F\left(x - \frac{1}{n}\right) \to F(x)$$

(By the def of continuity — more on this later)

**Fact.** If $g\colon \mathbb{R} \to \mathbb{R}$ is continuous, then

1. $X_n \xrightarrow{d} X \implies g(X_n) \xrightarrow{d} g(X)$
2. $X_n \xrightarrow{p} X \implies g(X_n) \xrightarrow{p} g(X)$

Remark: This fact is called the **continuous mapping theorem**

Example. If $\alpha$ is constant and $X_n \xrightarrow{d} X$, then

- $X_n + \alpha \xrightarrow{d} X + \alpha$
- $\alpha X_n \xrightarrow{d} \alpha X$
- etc.

# The Central Limit Theorem

Let $\{X_i\}_{i=1}^{\infty} \overset{\text{IID}}{\sim} F$ with

- $\mu := \mathbb{E}\left[X_i\right] = \int x F(dx)$
- $\sigma^2 := \text{var}[X_i] = \int (x-\mu)^2 F(dx)$, assumed finite

**Fact.** In this setting we have

$$\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{as} \quad n \to \infty$$

- $\bar{X}_n := \frac{1}{n} \sum_{i=1}^n X_i$
- $\xrightarrow{d}$ means the cdf of LHS $\to$ weakly to the $N(0, \sigma^2)$ cdf

Proof: Omitted

Alternative version: Under the same conditions we have

$$\sqrt{n}\left\{\frac{\bar{X}_n - \mu}{\sigma}\right\} \xrightarrow{d} N(0,1)$$

To see this let $Y \sim N(0, \sigma^2)$, so that $\sqrt{n}(\bar{X}_n - \mu) \xrightarrow{d} Y$

Applying the continuous mapping theorem gives

$$\sqrt{n}\left\{\frac{\bar{X}_n - \mu}{\sigma}\right\} \xrightarrow{d} \frac{Y}{\sigma}$$

Clearly $Y/\sigma$ is normal, with

$$\mathbb{E}\left[\frac{Y}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}[Y] = 0 \quad \text{and} \quad \text{var}\left[\frac{Y}{\sigma}\right] = \frac{1}{\sigma^2}\text{var}[Y] = 1$$

Discussion: The CLT tells us about distribution of $\bar{X}_n$ when

- sample is IID
- $n$ large

Informally,

$$\sqrt{n}(\bar{X}_n - \mu) \approx Y \sim N(0, \sigma^2)$$

$$\therefore \quad \bar{X}_n \approx \frac{Y}{\sqrt{n}} + \mu \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Thus, $\bar{X}_n$ approximately normal, with

- mean equal to $\mu$, and
- variance $\to 0$ at rate proportional to $1/n$

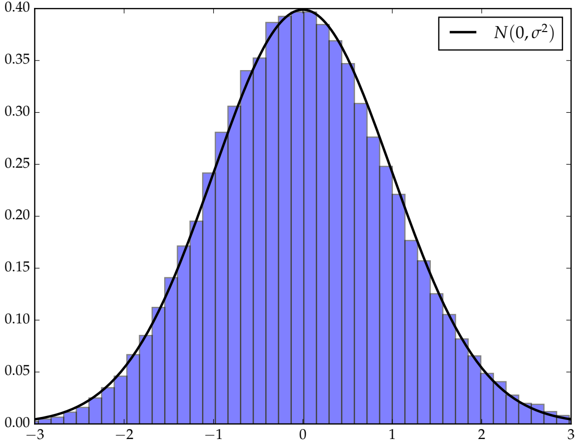
# Illustrating the CLT

We can illustrate the CLT with simulations by

1. choosing an arbitrary cdf $F$ for $X_n$ and a large value for $n$
2. generating independent draws of $Y_n := \sqrt{n}(\bar{X}_n - \mu)$
3. using these draws to compute some measure of their distribution, such as a histogram
4. comparing the latter with $N(0, \sigma^2)$

We do this for

- $F(x) = 1 - e^{-\lambda x}$ (exponential distribution)
- $n = 250$


0.40
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
−3
−2
−1
0
1
2
3
$N(0, \sigma^2)$

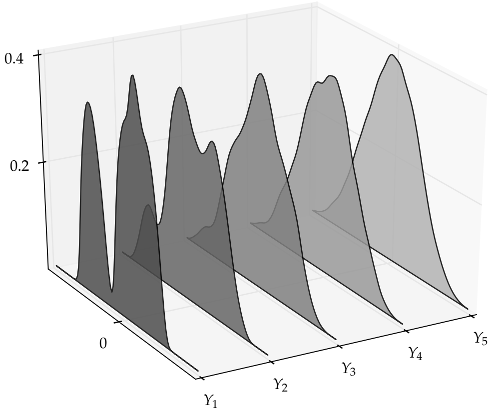
Another way we can illustrate the CLT:

Numerically compute the distributions of

1. $Y_1 = \sqrt{1}(\bar{X}_1 - \mu) = X_1 - \mu$
2. $Y_2 = \sqrt{2}(\bar{X}_2 - \mu) = \sqrt{2}(X_1/2 + X_2/2 - \mu)$
3. $Y_3 = \cdots$

The distribution of each $Y_n$ can be calculated once the distribution $F$ of $X_n$ is specified

The next figure shows these distributions for arbitrarily chosen $F$


0.4
0.2
0
$Y_1$
$Y_2$
$Y_3$
$Y_4$
$Y_5$

# Conditional Expectation

Let $X$ and $Y$ be two random variables

To economize on notation we overload the $p$ symbol by writing

- $p(x,y)$ for the joint density
- $p(y \mid x)$ for the conditional density of $y$ given $x$, etc.

Example. If on a computer we draw

1. $X \sim U[0,1]$
2. and then $Y \sim N(\mu, \sigma^2)$ with $\mu$ set to $X$

then

$$p(y \mid x) = p(y \mid X = x) = N(x, \sigma^2)$$

The **conditional expectation** of $Y$ given $X$ is then defined as

$$\mathbb{E}[Y \mid X] = \int y \, p(y \mid X) dy$$

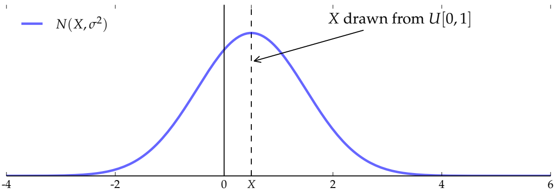
- Notation: Here and below, by convention, $\int := \int_{-\infty}^{\infty}$

The right hand side contains $X$, so it is a random variable!

In general,

- $\mathbb{E}[Y \mid X]$ is the "best predictor of $Y$ given $X$"
- A rule that maps $X$ into a prediction of $Y$
- And therefore a function of $X$
- And therefore random

Example. As before we draw $X \sim U[0,1]$ and then $Y \sim N(X, \sigma^2)$




We want a rule that maps $X$ to a prediction of $Y$

Intuition suggests that the best guess of $Y$ given $X$ is just $X$

Let's make sure this checks out

$$\mathbb{E}\left[Y \mid X\right] = \int y\, p(y \mid X) dy$$

For this case we saw that

$$p(y \mid x) = N(x, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-x)^2}{2\sigma^2} \right\}$$

$$\therefore \quad p(y \mid X) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-X)^2}{2\sigma^2} \right\}$$

$$\therefore \quad \mathbb{E}\left[Y \mid X\right] = \int y \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ \frac{-(y-X)^2}{2\sigma^2} \right\} dy$$

This is just the mean of $N(X, \sigma^2)$, which is $X$

Also intuitive: when $X$ and $Y$ are independent, $X$ is no help in predicting $Y$

- the same as predicting $Y$ with no information

Since $\mathbb{E}[Y]$ = best guess of $Y$ with no information, this suggests

$$\mathbb{E}[Y \mid X] = \mathbb{E}[Y]$$

The conjecture checks out too, since for this case we have

$$p(y \mid X) = \frac{p(y, X)}{p(X)} = \frac{p(y)p(X)}{p(X)} = p(y)$$

Hence

$$\mathbb{E}[Y \mid X] = \int y p(y \mid X) dy = \int y p(y) dy = \mathbb{E}[Y]$$

Sometimes we want to compute the conditional expectation of a function $f(X,Y)$ depending on both $X$ and $Y$

Example. Suppose that

- $Y$ is the payoff from a foreign asset, random
- $r(X)$ is an exchange rate, depending on some random $X$
- return in domestic currency is $f(X,Y) = r(X)Y$

What is the expectation of $f(X,Y)$ given $X$?

The general definition is

$$\mathbb{E}\left[f(X,Y) \mid X\right] = \int f(X,y)p(y \mid X)dy$$

For the preceding example this gives

$$\mathbb{E}\left[r(X)Y \mid X\right] = \int r(X)\, y\, p(y \mid X) dy$$

Since $r(X)$ doesn't depend on $y$ it can pass out of the integral

Hence

$$\mathbb{E}\left[r(X)Y \mid X\right] = r(X) \int y\, p(y \mid X) dy$$

That is,

$$\mathbb{E}\left[r(X)Y \mid X\right] = r(X)\mathbb{E}\left[Y \mid X\right]$$

This is a general rule — when conditioning on $X$, RVs depending only on $X$ can be passed out of the expectation

# The Multivariate Case

We can condition on $X_1, \ldots, X_K$ using

$$p(y \mid \mathbf{x}) = p(y \mid x_1, x_2, \ldots, x_K)$$
$$= p(y \mid X_1 = x_1, X_2 = x_2, \ldots, X_K = x_K)$$

Then we set

$$\mathbb{E}\left[Y \mid \mathbf{X}\right] := \int y\, p(y \mid \mathbf{X}) dy$$
$$= \int y\, p(y \mid X_1, X_2, \ldots, X_K) dy$$

- $\mathbf{X}$ can be a matrix: we condition on all $X_{ij}$ in $\mathbf{X}$

We can also extend the definition the case where $\mathbf{X}$ and $\mathbf{Y}$ are matrices

Given

$$\mathbf{Y} = \begin{pmatrix} Y_{11} & \cdots & Y_{1K} \\ \vdots & \vdots & \vdots \\ Y_{N1} & \cdots & Y_{NK} \end{pmatrix}$$

we set

$$\mathbb{E}\left[\mathbf{Y} \mid \mathbf{X}\right] = \begin{pmatrix} \mathbb{E}\left[Y_{11} \mid \mathbf{X}\right] & \cdots & \mathbb{E}\left[Y_{1K} \mid \mathbf{X}\right] \\ \vdots & \vdots & \vdots \\ \mathbb{E}\left[Y_{N1} \mid \mathbf{X}\right] & \cdots & \mathbb{E}\left[Y_{NK} \mid \mathbf{X}\right] \end{pmatrix}$$

We have provided some intuition for the following key facts

**Fact.** If $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ are random matrices and $\mathbf{A}$ and $\mathbf{B}$ are constant matrices, then, assuming conformability,

1. $\mathbb{E}\left[\mathbf{AX} + \mathbf{BY} \,|\, \mathbf{Z}\right] = \mathbf{A}\mathbb{E}\left[\mathbf{X} \,|\, \mathbf{Z}\right] + \mathbf{B}\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{Z}\right]$
2. If $\mathbf{X}$ and $\mathbf{Y}$ are independent, then $\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right] = \mathbb{E}\left[\mathbf{Y}\right]$
3. If $G(\mathbf{X})$ is a matrix depending only on $\mathbf{X}$, then
   - $\mathbb{E}\left[G(\mathbf{X})\,\mathbf{Y} \,|\, \mathbf{X}\right] = G(\mathbf{X})\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right]$
   - $\mathbb{E}\left[\mathbf{Y}\,G(\mathbf{X}) \,|\, \mathbf{X}\right] = \mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right] G(\mathbf{X})$
4. $\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{Z}\right]' = \mathbb{E}\left[\mathbf{Y}' \,|\, \mathbf{Z}\right]$
5. $\mathbb{E}\left[\mathbb{E}\left[\mathbf{Y} \,|\, \mathbf{X}\right]\right] = \mathbb{E}\left[\mathbf{Y}\right]$

No. 5 is called the **law of iterated expectations**

Let's just check that $\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] = \mathbb{E}\left[Y\right]$ in the scalar case

We have

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[Y \mid X\right]\right] &= \mathbb{E}\left[\int y\, p(y \mid X) dy\right] \\
&= \int \left[\int y\, p(y \mid x) dy\right] p(x) dx \\
&= \int y \left[\int p(y \mid x) p(x) dx\right] dy \\
&= \int y\, p(y) dy = \mathbb{E}\left[Y\right]
\end{aligned}$$

# New Topic

# ANALYSIS

# Motivation

We looked at linear systems carefully, but how about nonlinear systems?

- Solving nonlinear equations
- Optimization problems

How are these problems different?

What mathematics do we need to study them?

An example problem:

Let $f$ be a given nonlinear function

Does there exist an $\bar{x}$ such that $f(\bar{x}) = 0$?

Examples.

- $F$ is a profit function, $f = F'$, we're looking for stationary points of the profit function
- We want to solve an equation $g(\bar{x}) = y$ for $\bar{x}$
    - Set $f(x) = g(x) - y$

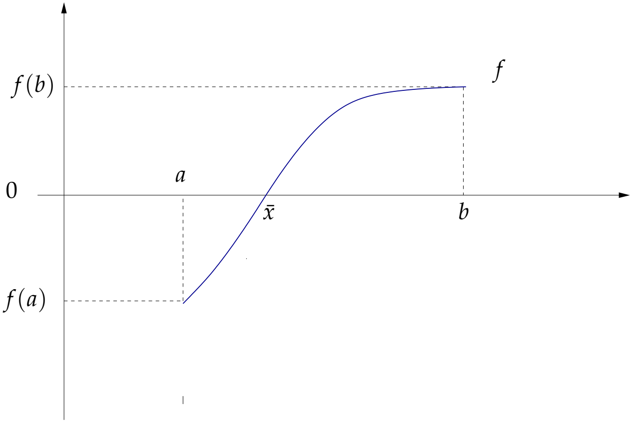



Figure : Existence of a root

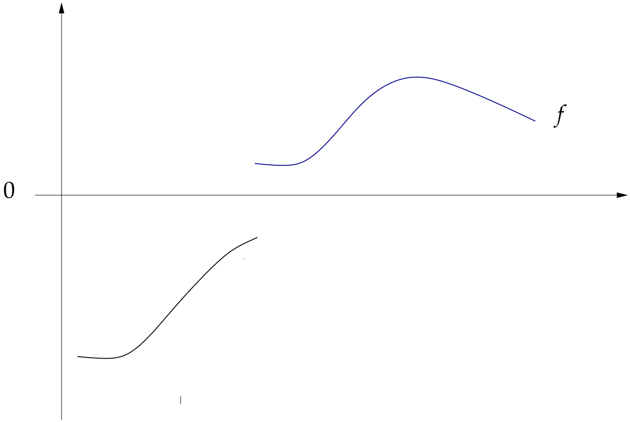



Figure : Non-existence of a root

One answer: a solution exists under certain conditions including continuity

Questions:

- So how can I tell if $f$ is continuous?
- Can we weaken the continuity assumption?
- Does this work in multiple dimensions?
- When is the root unique?
- How can we compute it?
- Etc.

These are typical problems in analysis